MATH 337
INTRODUCTION TO SOLITON THEORY
Homework set 5,6

April 18, 2012
Due May 3 , 2012

PROBLEMS.

1. a) Let Λ be an algebra of Laurent series. Let $L \in \Lambda$ be given by

$$L = \sum_{i=-\infty}^{\infty} u_i(x,t)\, p^i$$

where u_i are differentiable functions of x and t. A bracket $\{,\} : \Lambda \times \Lambda \to \Lambda$ is defined as

$$\{f,g\} = p\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}\right)$$

for all $f, g \in \Lambda$. Lax equation on Λ is given by

$$L_{,t} = \{A, L\}$$

where L is called the Lax operator and $A = (L^n)_{\geqslant 1}$. Here $()_{\geqslant 1}$ denotes that in the Laurent series of L consider only terms p^k with $k \geqslant 1$. Let $L = p + u(x,t) + v(x,t)\, p^{-1}$. Find the evolution equations satisfied by u and v when $n = 1, 2$.

2. Find a functional $A = \int\int L dx dt$ whose extremum is achieved by the solutions of the evolution equation $u_t - 10u^3\, u_x + u_{xxx} = 0$. **Hint**: Find the Lagrange function L such that the Euler-Lagrange equation, $E(L) = 0$, gives this evolution equation.

3. Let w_0 be a solution of the potential KdV equation. Starting from the solution w_0, let w_1, w_2 and w_3 be the solutions of the potential KdV equation

from the Backlund transformation corresponding to the Backlund parameters λ_1, λ_2 and λ_3 respectively. Prove that

$$w_4 = \frac{\lambda_1 w_1 (w_2 - w_3) + \lambda_2 w_2 (w_3 - w_1) + \lambda_3 w_3 (w_1 - w_2)}{\lambda_1 (w_2 - w_3) + \lambda_2 (w_3 - w_1) + \lambda_3 (w_1 - w_2)} \tag{1}$$

solves the potential KdV equation.