MATH 337
INTRODUCTION TO SOLITON THEORY
Homework set 3

March 22, 2012
Due April 3 , 2012

PROBLEMS.

1. Find three conservation laws for the mKdV equation

$$u_t - 6u^2\, u_x + u_{xxx}$$

where $x \in \mathbb{R}$ which involve u,u^2, and u^4, respectively.

2. Show that , if $u(x,t)$ is a solution of the nonlinear Schrodinger equation

$$iu_t + u_{xx} + \nu u|u|^2 = 0$$

where $x \in \mathbb{R}$ and ν is a real constant, then $\int_{\infty}^{\infty} |u|^2 dx$, $\int_{\infty}^{\infty} (u^* u_x - u u_x^*) dx$, $\int_{\infty}^{\infty} (|u_x|^2 - \frac{1}{2}\nu|u|^4) dx$, are constants of motion (The (*) denotes complex conjugate)

3. Given the Sine-Gordon equation in the form

$$\phi_{xt} = \sin\phi$$

verify that the following conservation laws

$$(\frac{1}{2}\phi_t^2)_x - (1 - \cos\phi)_t = 0,$$

$$(1 - \cos\phi)_x - (\frac{1}{2}\phi_x^2)_t = 0,$$

$$(\frac{1}{4}\phi_x^4 - \phi_{xx}^2)_t + (\phi_x^2 \cos\phi)_x = 0$$

4. *Frechet Derivative*. The Frechet derivative, $\frac{\delta F}{\delta u}$, of the operator $F\{u\}$, 's defined as

$$\int_{-\infty}^{\infty} v\, \frac{\delta F}{\delta u} dx = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \int_{-\infty}^{\infty} F(u + \epsilon v) dx$$

for all continuous v. Show that, if $F(u) = f(u, u_x, u_{xx}, \cdots)$, then $\frac{\delta F}{\delta u}$ corresponds to the Euler-Lagrange operator, i.e.,

$$\frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} - \frac{d}{dx}\frac{\partial f}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xx}} - \cdots$$

Hence verify that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{\delta}{\delta u}\,(u^3 + \frac{1}{2}u_x^2)\right\}$$