MATH 543: METHODS OF APPLIED MATHEMATICS I

Second Midterm Exam Solutions:

05 December 2001 Wednesday 18.00-20.00SAZ:13

QUESTIONS:

(25)**1**. Solve $u'' = f(x)$ with $u(0) = \alpha$ and $u'(a) = \beta$ by using the Green's function method.

Solution: The general solution should be

$$u(x) = cu_1(x) + bu_2(x) + \int_0^a G(x, y)\, f(y) dy$$

where c and b are constants to be determined and u_1 and u_2 are the solutions of the homogenous equation. The fundamental set is $\{u_1 = 1, u_2 = x\}$. Hence letting

$$G(x, y) = \begin{cases} a_1 + a_2\, x & x \leq y \\ a_2 + b_2\, x & x \geq y \end{cases}$$

and using the homogenous boundary conditions and the continuity conditions we obtain

$$G(x, y) = \begin{cases} -x & x \leq y \\ -y & x \geq y \end{cases}$$

The constants c and b are given by

$$u(0) = c = \alpha, \quad u'(a) = b = \beta$$

Hence the complete solution is

$$u(x) = \alpha + \beta x - \int_0^x y f(y) dy - x \int_x^a f(y) dy$$

(25)**2.** (a) Find the complete solution of the differential equation $zu'' + u' - u = 0$. First obtain the regular (analytic) and the singular solution. Solve also the initial value problem $u(0) = \alpha$ where α is any real constant.

Solution: If you solve the indical equation you find $r_1 = r_2 = 0$. Firs find $C_n(r)$. Letting $u(z) = \sum_{n=0}^{\infty} C_n\, z^{n+r}$ we find

$$C_n(r) = \frac{C_0}{[(n+r)(n+r-1)\cdots(r+1)]^2}$$

(a) Analytic solution: $r = 0$ then $u_1(z) = \sum_{n=0}^{\infty} C_n(0) z^n$

$$u(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n!]^2}$$

Here we took $C_0 = 1$

(b) For the second solution we use the formula

$$u_2 = u_1(z) \ln z + \sum_{n=0}^{\infty} C_n'(0)\, z^n$$

where

$$C_n(0) = -2C_n(0)\, \varphi_n, \quad \varphi_n = \sum_{k=1}^{k=n} \frac{1}{k}$$

Hence we have

$$u(z) = u_1 \ln z - 2 \sum_{n=1}^{\infty} \frac{\varphi_n z^n}{[n!]^2}$$

(c) The general solution of a boundary value problem in the neighborhood of the singular point $z = 0$ should be $u(z) = au_1 + bu_2$ since $u(0) = \alpha$ (finite) the $b = 0$ and $a = 1$. Then the solution of such a boundary value problem is $u = \alpha\, u_1(z)$

(25)**3.** Hypergeometric DE equation is defined as $z(1-z)u'' + [c - (1+a+b)z]u' - abu = 0$ and the solution analytic about the regular singular point

$z = 0$ is known as the hypergeometric function $F(a, b; c; z)$. which has a series solution

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

where $\Gamma(x)$ is the gamma function.

(a) Confluent hypergeometric function $\Phi(a, c; z)$ defined as

$$\Phi(a, c; z) = \lim_{b \to \infty} F(a, b; c; z/b)$$

Prove that $u = \Phi(a, c; z)$ satisfies $zu'' + (c - z)u' - au = 0$

(b) Find the series (expansion) solution of $\Phi(a, c; z)$ about $z = 0$ (analytic solution about $z = 0$).

(c) Prove that

$$\frac{d^n}{dz^n}\Phi(a, c; z) = \frac{\Gamma(a+n)\,\Gamma(c)}{\Gamma(c+n)\,\Gamma(a)}\,\Phi(a+n, c+n; z)$$

Solution: (a) Let $z = \xi/b$ then the hypergeometric equation becomes

$$\xi(1 - \frac{\xi}{b})u_{\xi\xi} + [c - \xi - (1+a)\frac{\xi}{b}]u_{\xi} - au = 0$$

Define $v(z) = \lim_{b\to\infty} u(z) = \Phi(a, c; z)$ and letting $b \to \infty$ in the above equation we find confluent hypergeometric equation

$$\xi v_{\xi\xi} + (c - \xi)v_{\xi} - av = 0$$

(b)

$$\Phi(a, c; z) = \lim_{b\to\infty} F(a, b, c; z/b)$$
$$= \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} \lim_{b \to]infty} \frac{\Gamma(b+n)}{\Gamma(b)b^n}$$

Here you recognize that the limit and sum are inetrcahneged safely becouse the series expansion of the hypergeometrice function $F(a, b, c; z)$ is uniformly convergent for $|z| < 1$. The limit inside the sum is equal to unity

$$\lim_{b\to\infty} \frac{\Gamma(a+n)}{\Gamma(b)b^n} = 1$$

Hence the series expansion of the confluent hypergeometric function about the regular singular point $z = 0$ is found as

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

(c) One can use mathematical induction to prove this identity, but here I will use a different proof. Use the series expansion about $z = 0$ given above and differentaite both sides m times then we obtain

$$\frac{d^m}{dz^m} \Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=m}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^{n-m}}{(n-m)!}$$

Here we started the sum from m. All values of the sum corresponding to $n < m$ vanish. Changing the integration variable $n = m + k$ the one proves the identity.

(25)**4**. Solve $u'''' = f(x)$ with $u(0) = 0,\ u'(0) = 2u'(1),\ u(1) = 0,\ u''(0) = 0$

Solution: This is a fourth order difefrential equation but the application of the Green's function technique is the same as we studied in class. We have

$$u^4(x) = f(x), \quad \frac{d^4}{dx^4} G^(x, y) = \delta(x-y)$$

hence

$$G(x, y)\, u^4(x) - u(x)\, G^4(x, y) = f(x)\, G(x, y) - u(x) - \delta(x-y)$$

integrating over $[0,1]$ we get

$$u(y) = \int_0^1 G(x,y)\, f(x) dx$$

and the surface term

$$\int_0^1 [(Gu^3 - uG^3)' - (G'u^3 - u'G^3)] dx = 0$$

This can be integrated and we find that

$$G(1,y)\, u^3(1) - G(0,y)u^3(0) - G'(1,y)u^2(1) + u'(1)[G^2(1,y) - 2G^2(0,y) = 0$$

which leads to the adjoint boundary conditions

$$G(0,y) = 0, \quad G(1,y) = 0, \quad G'(1,y) = 0, \quad G^2(1,y) = 2G^2(0,y)$$

The fundamental set for this problem is $\{1, x, x^2, x^3\}$. Hence letting

$$G(x,y) = \begin{cases} U_< & x \le y \\ U_> & x \ge y \end{cases}$$

where

$$U_< = a_1 + a_2 x + a_3 x^2 + a_4 x^3, \quad U_> = b_1 + b_2 x + b_3 x^3 + b_4 x^3$$

Here a_i, b_i are constants (wrt x) to be determined. In addition to the above adjoint boundary conditions there are also the following continuity conditions: $\frac{d^k}{dx^k} G(x,y)$ are continuous for $k = 0, 1, 2$ and

$$G^3(x,y)|_- - G^3(x,y)|_+ = 1$$

Using all the 8 conditions one finds (I was not expecting you to derive the below expressions. The above information (if correct) is enough form me.

$$U_< = \frac{1}{10} y(2 + 3y^2)x - \frac{1}{2} y^2 x + \frac{1}{10} y(1 - y^2)x^2 + \frac{1}{5}(y - \frac{1}{6} y^3)x^3 - \frac{1}{6} x^3$$
$$U_> = -\frac{1}{6} y^3 + \frac{1}{10} y(2 + 3y^2)x - \frac{1}{10} y(4 + y^2)x^2 + \frac{1}{5}(y - \frac{1}{6} y^3)x^3$$