MATH 240: FROBENIUS METHOD

The following theorem summarizes the Frobenius method:

Theorem 1. *Suppose that the differential equation $Lu = 0$, where L is a second order differential operator, has a regular singular point at $z = z_0$ with the roots r_1 and r_2 of the indical equation. There are three possible cases: (Assuming $Re(r_1) > Re(r_2)$)*

1. $r_1 - r_2 \neq$ *an integer*
2. $r_1 - r_2 = N$ *(a non-negative integer) and recursion relation is consistent*
3. $r_1 - r_2 = N$ *(a non-negative integer) and recursion relation is not consistent.*

Then, in the first two cases, there exists a bases of $\{u_1, u_2\}$ of solutions of $Lu = 0$. These solutions are of the form

$$\begin{aligned} u_1(z) &= (z - z_0)^{r_1} \sum_{k=0}^{\infty} C_k^1 (z - z_0)^k, && (1) \\ u_2(z) &= (z - z_0)^{r_2} \sum_{k=0}^{\infty} C_k^2 (z - z_0)^k && (2) \end{aligned}$$

and the third case , the bases $\{u_1, u_2\}$ is of the following form

$$\begin{aligned} u_1(z) &= (z - z_0)^{r_1} \sum_{k=0}^{\infty} C_k (z - z_0)^k, && (3) \\ u_2(z) &= C u_1(z) \ln(z - z_0) + (z - z_0)^{r_2} \sum_{k=1}^{\infty} B_k (z - z_0)^k && (4) \end{aligned}$$

where the power series about $(z - z_0)$ are convergent in a neighborhhod of $z = z_0$

A method when $r_1 - r_2 = N$
Let $z = z_0$ be a regular singular point of differential equation

$$Lu = u'' + p(z)\,u' + q(z)u = 0, \tag{5}$$

where

$$A(z) = (z - z_0)\,p(z), \quad B(z) = (z - z_0)^2\,q(z) \tag{6}$$

are analytic in a neighborhood D of $z = z_0$ hence $A(z)$ and $B(z)$ are analytic in D then

$$A(z) = \sum_{k=0}^{\infty} a_k\,(z - z_0)^k, \quad B(z) = \sum_{k=0}^{\infty} b_k\,(z - z_0)^k, \tag{7}$$

Using the anzats

$$u(z) = (z - z_0)^r \sum_{k=0}^{\infty} C_k\,(z - z_0)^k, \tag{8}$$

in the DE we get

$$\lambda_0(n + r)\,C_n + \sum_{m=0}^{n-1} [(m + r)a_{n-m} + b_{n-m}]\,C_m, \tag{9}$$

This is the recursion relations mentioned in the above theorem 1. Here

$$\lambda_0(r) = r(r - 1) + r\,a_0 + b_0 \tag{10}$$

If the indical equation is not imposed $\lambda_0(r) \neq 0$ and the remaining

$$C_1(r), C_2(r), \cdots, C_n(r) \tag{11}$$

are determined through the recursion relation. Since $n = 0$ term is not set to zero then the differential equation is not satisfied but becomes

$$Lu = C_0\,\lambda_0(r)\,(z - z_0)^{r-2} \tag{12}$$

where C_0 is the undetermined constant. The solution of this equation is

$$u(r, z) = (z - z_0)^r \sum_{k=0}^{\infty} C_k(r)\,(z - z_0)^k, \tag{13}$$

In the case 3 , when the difference of the indices is a nonegative number and recursion realtions are inconsistent the the usual anzats fails. To find the correct anzats and a method of solution we use (12) and (13).

$r_1 = r_2$ **case**
Eq.(12) becomes

$$Lu = C_0\,(r - r_1)^2\,(z - z_0)^{r-2}, \tag{14}$$

Taking the derivative of both sides wrt r and letting $r = r_1$ we obtain the second solution as

$$u_2(z) = \frac{du(r,z)}{dr}|_{r=r_1}, \tag{15}$$

Using the form of $u(r,z)$ given in (13) we obtain

$$u_2(z) = u_1(z)\,\ln(z - z_0) + (z - z_0)^{r_1} \sum_{k=0}^{\infty} C_k'(r_1)\,(z - z_0)^k \tag{16}$$

where $C_k'(r_1) = \frac{d}{dr}\,C_k(r)\,|_{r=r_1}$

$r_1 - r_2 = N$ **a positive integer case**
Eq.(12) becomes

$$Lu = C_0(r - r_1)(r - r_2)\,(z - z_0)^{r-2} \tag{17}$$

Multiplying both sides by $r - r_2$ and taking derivatives of both sides wrt r and taking limit as r goes to r_2 we obtain

$$u_2(z) = \frac{d}{dr}\,[(r - r_2)\,u(r,z)] \;\text{ at } r = r_2 \tag{18}$$

Using the expression (13) for $u(r,z)$ we obtain

$$u_2(z) = C u_1(z)\,\ln(z - z_0) + (z - z_0)^{r_2} \sum_{k=0}^{\infty} \tilde{C}_k\,(z - z_0)^k \tag{19}$$

where

$$\tilde{C}_k = \frac{d}{dr}[(r - r_2)\, C_k(r)]\,|_{r=r_2}$$

Problems

1. An example for the case $r_1 - r_2 = N$ with consistent recursion realtions. Solve $z^2u'' + (z^2 + z)u' - u = 0$ about $z = 0$ (solved in Class)

2. An example for the case $r_1 = r_2$ with inconsistent recursion relarions. Solve $z^2u'' + zu' + z^2u = 0$ about $z = 0$. You can find the solution u_1 by using the recursion relations. This solution is known as the Bessel function of order 0,

$$u_1(z) = J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(k!)^2}$$

The second solution $u_2(z)$ is found by using (16). Prove that it takes the form

$$u_2(z) = J_0(z)\,\ln z + \sum_{k=0}^{\infty} \varphi(k) \frac{(z/2)^{2k}}{(k!)^2}$$

where

$$\varphi(k) = \begin{cases} \sum_{m=1}^{k} \frac{1}{m}, & \text{for } k = 1, 2, \cdots \\ 0, & \text{for } (k = 0) \end{cases} \tag{20}$$

3. An example for the case $r_1 - r_2 = N$ with inconsistent recursion relations. Solve $z\,u'' - u = 0$ about $z = 0$. Again $u_1(z)$ will be found by the standard method

$$u_1(z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!(k+1)!}$$

And the second solution is found by using the formula (19)

$$u_2(z) = u_1\,\ln z + 1 - \sum_{k=1}^{\infty} \frac{\varphi(k) + \varphi(k-1)}{k!(k-1)!}\, z^k$$

4. Find the solutions of the Bessel's equation $z^2u'' + zu' + (z^2 - p^2)u = 0$. Solutions of this equation are known as the Bessel's functions of order p. Solve this equation for $p =$ not integer and for $p =$ integer.

5. Show that for the differential equation $z\,u'' + 3u' + 4zu = 0$ the condition $u(0) = 1$ determines a unique solution, and that $u'(0)$ cannot also be prescribed. Determine this solution.

6. Use the method of Frobenius to obtain the general solution of each of the following differential equations, valid near $z = 0$:

(1) $z^2u'' - 2zu' + (2 - z^2)u = 0$
(2) $(z - 1)u'' - zu' + u = 0$
(3) $2zu'' + (1 - 2z)u' - u = 0$
(4) $z^2u'' + zu' + (z^2 - \frac{1}{4})u = 0$
(5) $zu'' - u' + 4z^3u = 0$
(6) $zu'' + 2u' + zu = 0$
(7) $z(1 - z)u'' - 2u' - 2 + 2u = 0$

7. Determine the two values of the constant α for which it is true that all solutions of the equation $zu'' + (z - 1)u' - \alpha u = 0$ are regular at $z = 0$, and obtain the general solution in each of these cases

8. (a) Show that the equation $zu'' + u' - u = 0$ possesses equal indices $r_1 = r_2 = 0$ at $z = 0$
(b) Obtain the regular solution $u_1(z)$
(c) Assume the second solution of the form $u_2(z) = Cu_1(z)\ln z + v(z)$ where $C \neq 0$. Find $v(z)$.

9. (a) Show that the equation $zu'' - zu' - u = 0$ possesses indices $r_1 = 1, r_2 = 0$ at $z = 0$.
(b) Obtain the regular solution $u_1(z)$
(c) Assume the general form for the second solution as $u_2(z) = Cu_1\ln z + v(z)$, where $C \neq 0$. Find $v(z)$.