Optional Homework

MATH 240: GREEN'S FUNCTIONS: Related to the problems 139/32, 178/27, 659/28-34

Generalized Green's Identity

Let L_x be a differential operator and u , v are some functions in $L^2_w(a,b)$ then the *adjoint operator* $L^\dagger$ is defined through the *Lagrange identity*

$$w[\bar{v}L_x u - u\,\overline{\{L_x^\dagger v\}}] = \frac{d}{dx}Q(u,\bar{v}), \tag{1}$$

where $w > 0$ is the weight function and $Q(u,\bar{v})$ has in general the following form

$$Q(u,\bar{v}) = Au\bar{v} + B\bar{v}\frac{du}{dx} + Cu\frac{d\bar{v}}{dx} + D\frac{du}{dx}\frac{d\bar{v}}{dx}, \tag{2}$$

Integrating the above identity over the interval $[a,b]$ we get the *generalized Green's identity*

$$\int_a^b w(x)\,[\bar{v}\,L_x u - u\,\overline{\{L_x^\dagger\, v\}}]dx = Q(u(b),\bar{v}(b)) - Q(u(a),\bar{v}(a)), \tag{3}$$

The right hand side of this equation is called the *surface term.*

Adjoint Boundary Conditions:

Let the function u satisfy the boundary conditions

$$B_1(u) = \alpha_1 u(a) + \beta_1 u'(a) + \gamma_1 u(b) + \delta_1 u'(b) = 0, \tag{4}$$

$$B_2(u) = \alpha_2 u(a) + \beta_2 u'(a) + \gamma_2 u(b) + \delta_2 u'(b) = 0, \tag{5}$$

Here $\alpha_i, \beta_i, \gamma_i$, and δ_i are some given constants. If in the above *generalized Green's identtiy* (3) the function u satisfies the above boundary condtions (4) and (5) then the function v satisfies the *adjoint boundary conditions* when the *surface term vanishes*. Hence for future purposes we can define the *Green's identity*

$$\int_a^b w[\bar{v}\, L_x u - u\, \overline{\{L_x^\dagger\, v\}}]dx = 0, \tag{6}$$

for the functions u satsfying the boundary conditions (4) and (5) and v satisfying the *adjoint boundary condtions*

Self Adjoint Operators:

We sall be interested in second order differential operators

$$L_x u(x) = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u, \tag{7}$$

where the coefficients a, b and c are some functions of x. Here we assume that $Re(a) > 0$. It is easy to show that the adhoint operator is given by

$$L^\dagger v = \frac{d^2}{dx^2}(\bar{a}\, v) - \frac{d}{dx}(\bar{b}v) + \bar{c}v, \tag{8}$$

and the function Q in the surface term is found as

$$Q(u, \bar{v}) = a\bar{v}\frac{du}{dx} - u\frac{d}{dx}(a\bar{v}) + bu\bar{v}, \tag{9}$$

If the coefficient functions are real and

$$b(x) = \frac{da(x)}{dx}, \tag{10}$$

Then $L_x^\dagger = L$, becomes a self-adjoint operator and the Q term becomes.

$$Q(u, \bar{v}) = a(\bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}), \tag{11}$$

With a suitable weight factor all second order differential operators are self-adjoint

Theorem 1. *Every linear second order differential operator with real coefficients is self adjoint provided the weight function $w(x)$ is choosen properly*

$$w(x) = \frac{p}{a}, \qquad \frac{p'}{p} = \frac{b(x)}{a(x)}, \tag{12}$$

where the diferential operator takes the form

$$L_x u = \frac{1}{w}\left(\frac{d}{dx}\, p(x)\frac{du}{dx}\right) + cu, \tag{13}$$

and the function Q is given by

$$Q(u, \bar{v}) = p(x)\,(\bar{v}\frac{du}{dx} - u\frac{d\bar{v}}{dx}), \tag{14}$$

Problems:

1. Prove that the following boundary conditions on $u(x)$, L_x defines a Hermitian differential operator L

(i) $u(a) = u(b) = 0$ (Dirichlet conditons),

(ii) $\frac{du}{dx}(a) = \frac{du}{dx}(b)$, (Neumann conditions),

(iii) $\alpha\, u(a) - \frac{du}{dx}(a) = \beta u(b) - \frac{du}{dx}(b)$, ($\alpha, \beta$ real) (general unmixed conditions),

(iv) $u(a) = u(b)$ and $\frac{du}{dx}(a) = \frac{du}{dx}(b)$, (periodic conditions).

2. Determine the formal adjoint of each of the operators in (a) throuh (d) below (i) os a differential operator, and (ii) as an opeartor, that is, including the boundary conditions (Determine the functions spaces of L and $L^\dagger$). Which opeartors are formally self-adjoint? Which opeartors are self- adjoint?

(a) $L_x = \frac{d^2}{dx^2} + 1$ in $[0, 1]$ with $u(0) = u(1) = 0$,

(b) $L_x = \frac{d^2}{dx^2}$ in $[0, 1]$ with $u(0) = u'(0) = 0$,

(c) $L_x = \frac{d}{dx}$ in $[0, \infty)$ with $u(0) = 0$,

(d) $L_x = \frac{d^3}{dx^3} - \sin x \frac{d}{dx} + 3$ in $[0, \pi]$ with $u(0) = u'(0) = 0$ and $u''(0) - 4u(\pi) = 0$

Boundary value problems

Let u satisfy the inhomogenous differential equation with homogenous boundary conditions

$$L_x\, u(x) = f(x), \quad B_1(u) = 0, \quad B_2(u) = 0, \tag{15}$$

where $f(x)$ is any integrable function. The function v satisfying the adjoint differential and the homogenous adjoind boundary conditions

$$L_x^{\dagger}\, v(x) = h(x), \quad adB_1(v) = 0, \quad adB_2(v) = 0, \tag{16}$$

Green's Functions:
Green's functions $G(x, y)$ and $g(x, y)$ of the above boundary value problems are, respectively given by

$$L_x\, G(x, y) = \frac{\delta(x - y)}{w(x)}, \tag{17}$$

$$L_x^{\dagger}\, g(x, y) = \frac{\delta(x - y)}{w(x)}, \tag{18}$$

using the Gren's identity for the function u, v, G, and g pairwise (u, v), (u, g), (v, G) and (G, g) we obtain the following important identities

$$u(x) = \int_a^b w(y)\bar{g}(y, x)\, f(y)dy,, \tag{19}$$

$$v(x) = \int_a^b w(y)\bar{G}(y, x)\, h(y)dy, \tag{20}$$

and

$$< f, g > = < G, h >, \quad G(x, y) = \bar{g}(y, x), \tag{21}$$

Since the solutions are given in (19) and (20), then the boundary value problems (15) and (16) reduces to the determinattion of the Green's functions G and g

Second order operators and Green's functions:
For a second order opeartors with real coefficients we have

$$
\begin{aligned}
\frac{d}{dx}(p(x)\frac{du}{dx}) + cwu &= f(x), \quad B_1(u) = 0, \quad B_2(u) = 0, && (22)\\
\frac{d}{dx}(p(x)\frac{dv}{dx}) + cwv &= h(x), \quad adB_1(v) = 0, \quad adB_2(v) = 0, && (23)
\end{aligned}
$$

and the Green's functions satisfy the following equations

$$
\begin{aligned}
\frac{d}{dx}(p(x)\frac{dG(x,y)}{dx}) + cwG(x,y) &= \delta(x-y), && (24)\\
\frac{d}{dx}(p(x)\frac{dg(x,y)}{dx}) + cwg(x,y) &= \delta(x-y), && (25)
\end{aligned}
$$

Since the Greens's fuctions satisfy the homogenous equation ($L_x G(x,y) = 0$) for $x \neq y$ the we have

$$
G(x,y) = \begin{cases} c_1 u_1(x) + c_2 u_2(x) & a \leq x < y \\ d_1 u_1(x) + d_2 u_2(x) & x < y \leq b \end{cases} \tag{26}
$$

where u_1 and u_2 are the independent solutions of the homogenous equation $L_x u = 0$ and c_1, c_2, d_1, d_2 are constants (wrt x) to be determined by the following conditions on G. First two are the homogenous boundary conditions and the rest two are the continuity of G and jump condition for the derivative of G

$$
\begin{aligned}
B_1(G) = 0, \quad B_2(G) = 0, && (27)\\
G(x,y)|_{x=y_+} = G(x,y)|_{x=y_+}, && (28)\\
\frac{dG(x,y)}{dx}|_{x=y_-} - \frac{dG(x,y)}{dx}|_{x=y_-} = \frac{1}{p(y)}, && (29)
\end{aligned}
$$

We have the following important teorem.

Theorem 2. *Consider the boundary value problem $L_x u = f(x)$ with the homogenous boundary conditons $B_1(u) = 0$ and $B_2(u) = 0$. Here we assume* *that $L_x$is a second order differential operator. Provided that the homogenous equation has no notrivial solution satisfying the above boundary conditions the Green's function associated with the boundary value problem exist and unique. The solution is given by*

$$u(x) = \int_a^b w(y)\, G(x,y)\, f(y)dy \tag{30}$$

Problems:
3. Let $B_1(u) = 0$ and $B_2(u) = 0$ are defined at $x = a$ and $x = b$ respectively. Solve the Green's function defined in (26).
Solution: The continuity condtions (28) and (29) are solved easily (see DK page 281)

$$G(x,y) = \begin{cases} \frac{U_<(x)U_>(y)}{p(y)W} & a \le x < y \\ \frac{U_<(y)U_>(x)}{p(y)W} & y < x \le b \end{cases} \tag{31}$$

where $W = U_< U_>' - U_> U_<'$ is the Wronksian of $U_<(x) = u_1(x) - \alpha\, u_2(x)$ and $U_>(x) = u_1(x) - \beta u_2(x)$. Here u_1 and u_2 are independent solutions of the homogenous DE (the fundamenatal set). Prove that $p(x)W(U_<, U_>)$ is constant for all x. The boundary conditions (27) mus also be satisfied, hence

$$B_1(U_<) = B_1(u_1) - \alpha B_1(u_2) = 0,$$
$$B_2(U_>) = B_2(u_1) - \beta B_2(u_2) = 0$$

Hence α and β are determined. As we discussed above the product pW is assumed to be a nonvanishing constant. If it is zero, then $U_<(x) = \sigma U_>(x)$, where σ is an arbitaray constant. Using the definitions of $U_<$ and $U_>$ and α and β found above this relation gives ,

$$B_1(u_1 - \rho u_2) = 0, \quad B_2(u_1 - \rho u_2) = 0$$

We obtain this by letting $\sigma = \rho \frac{B_2(u_2)}{B_2(u_1)}$, where ρ is another constant. The above relation implies the existance of a nontrivial solution $u_1 - \rho u_2$ of the homogenous DE satisfying the homogenous boundary condions, but this is contardiction with our assumtion.

4. Solve the Green's function when both boundary conditions are given at the same point $x = a$. As an example take for instantce $u(a) = 0$ and $u'(a) = 0$.