MATH 240
DIFFERENTIAL EQUATIONS
First Midterm Exam

June 28 , 2002
Friday 13.40-15.30, SAZ-04-SAZ-8

corrected version

QUESTIONS:

[25]**1.** An initial value problem is given as follows:

$$\frac{dy}{dt} = a\,y + b\,y^2, \quad y(0) = y_0$$

where $a > 0,\ b > 0$ and y_0 is any real number.
(a) Find the equilibrium solutions and investigate their stability
(b) Solve the initial value problem.

solution: **(a)**. $f(t, y) = y(a + by)$. Equilibrium solutions are $y = 0$ and $y = -\frac{a}{b}$. Thier stability can be studied by letting (i) $y = u$ where u is considered as a small perturbation ($u^2 \equiv 0$), then we obtain $u' = au$ which has the solution $u = ce^{at}$. Since y goes to infinity as $t \to \infty$ then the equilibrium solution $y = 0$ is not **stable**. (ii) $y = -\frac{a}{b} + u$ then we get $u' = -at$ which has the solution $u = ce^{-at}$. Since y goes to zero as $t \to \infty$ then the equilibrium solution $y = -\frac{a}{b}$ is **stable**.
(b). The above differential equation can be written as

$$\frac{1}{a}(\frac{1}{y} - \frac{b}{a + by})dy = dt$$

The the solution can be found easily as

$$y = \frac{aC}{e^{-at} - bC}$$

where C is the integration constant. By the use of the initial condition $y(0) = y_0$ we find that

$$C = \frac{y_0}{by_0 + a}$$

and the solution of the initial value problem becomes

$$y = \frac{ay_0}{-by_0 + (by_0 + a)e^{-at}}$$

[25]**2**. (a) Solve the following initial value problem.

$$\frac{dy}{dt} = \frac{y - 2t}{2y - t}, \quad y(0) = y_0$$

(b) Deteremine the interval where the solution exists.

solution: Write the differential equation in the form

$$(2t - y)dt + (2y - t)dy = 0$$

which is an exact differential equation. $M = 2t - y, \; N = 2y - t$

$$M_y = -1, \quad N_t = -1$$

Hence $M = \psi_t$ and $N = \psi_y$ which leads to

$$\psi(t, y) = y^2 + t^2 - ty = c$$

where c is a constant to be determined by the initial condition $y(0) = y_0$. We find that $c = y_0^2$, then we get

$$y^2 + t^2 - ty = y_0^2$$

When we solve it we get

$$y = \frac{1}{2}\,[t + \sqrt{4y_0^2 - 3t^2}], \quad y_0 > 0, \tag{1}$$

$$y = \frac{1}{2}\,[t - \sqrt{4y_0^2 - 3t^2}], \quad y_0 < 0 \tag{2}$$

(3)

The interval of definition is: $4y_0^2 - 3t^2 > 0$ or

$$-\frac{2}{\sqrt{3}}|y_0| < t < \frac{2}{\sqrt{3}}|y_0|$$

Hence y_0 cannot be zero. If it is zero the solution becomes complex. Interval should be open. At the end points the solution becomes $y = t/2$ where the differential equation is not defined.

[25]**3**. Assume that a spherical raindrop evaporates at a rate proportional to the square of its surface area. If the radius is R_0 at the beginning ($t = 0$), and one our later has been reduced to $R_0/2$, find the expression for the radius of the raindrop at any time.

solution: From the question we understand that

$$\frac{dV}{dt} = kS^2$$

where V is the volume, S is the surface are of the sphere with radius R, and k is a constant. Since

$$V = \frac{4\pi}{3}R^3, \qquad S = 4\pi R^2$$

we find that

$$\frac{dR}{dt} = 4\pi k R^2$$

which has the solution

$$R(t) = -\frac{1}{C + 4\pi kt}$$

where C is the integration constant. The initial condition

$$R_0 = -\frac{1}{C}$$

and at $t = 1h$ we have $R = \frac{R_0}{2}$ this leads to

$$4\pi k R_0 = -1$$

then we get

$$R(t) = \frac{R_0}{1 + t}$$

[25]**4**. Let

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + (1 - a)y = 0,$$
$$y(0) = 0, \quad y'(0) = 1$$

where a is any real number. Solve the above initial value problem for all values of a { $(a < 0)$, $(a = 0)$, $(a > 0)$ }.

solution: The discriminant $\Delta = a$. Hence we have:
(i) $a > 0$, $\Delta >$) then $r_1 = -1 + \sqrt{a}$, $r_2 = -1 + \sqrt{a}$

$$y(t) = e^{-t}\,(C_1\, e^{\sqrt{a}t} + C_2\, e^{-\sqrt{a}t})$$

from the initial conditions

$$C_1 = \frac{1}{2\sqrt{a}}, \quad C_2 = -C_1$$

(ii) $a = 0$ $(\Delta = 0)$ then

$$y(t) = e^{-t}\,(C_1 + C_2 t)$$

with $C_1 = 0, \quad C_2 = 1$.
(iii) $a < 0$ ($\Delta < 0$, complex roots)

$$y(t) = e^{-t}\,[C_1\,\cos(\sqrt{a}t) + C_2\,\sin(\sqrt{a}t)]$$

with $C_1 = 0, \quad C_2 = \frac{1}{\sqrt{a}}$.