MATH 240
DIFFERENTIAL EQUATIONS
Second Midterm Exam

July 15 , 2002
Monday 14.40-16.30, SAZ-04-SAZ-8

corrected version

QUESTIONS: Solve only four problems Problems with solutions

[25]**1**. Let one of the solutions of the equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

be $(1+x)^2$ and the Wronskian of any two solutions of this equation be given as a constant everywhere. Find the general solution of

$$y'' + p(x)\,y' + q(x)\,y = 1 + x$$

solution: Since $W(y_1, y_2) = ce^{-\int p(x)dx} =$ constant then $p(x) = 0$. Since one solution is $(1+x)^2$ then inserting into the differential equation one finds that $q(x) = -\frac{2}{(1+x)^2}$. Hence we have

$$y'' - \frac{2}{(1+x)^2}y = 1 + x$$

Using the method of undetermined coefficients we obtain

$$y(x) = C_1\,(1+x)^2 + \frac{C_2}{(1+x)} + \frac{1}{4}(1+x)^3$$

[25]**2**. **(a)** Find a second linearly independent solution to the following equation if e^x is a solution for $x > 0$

$$xy'' + (1-2x)\,y' + (x-1)\,y = 0, \quad (x > 0)$$

(b). Find the solution of the following initial value problem

$$y'' + x\,y = x^2, \quad y(0) = 0, \quad y'(0) = 1$$

solution: **(a)**. Let $y(x) = e^x\, u(x)$. Insert it into the differential equation we get

$$xu'' + u' = 0$$

the solution of this differential equation is $\ln x$. Hence the general solution is given by

$$y(x) = C_1 e^x + C_2\, e^x\, \ln x$$

(b). $y(x) = x$ is a solution of the differential equation and the initial conditions. Due to the uniqueness theorem it is the solution of the IVP.

[25]**3**. Solve the following initial value problem (Legendre equation of order one)

$$(1 - x^2)\, y'' - 2x\, y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

solution: Since $x = 0$ is an ordinary point the we can make use of power series expansion

$$y(x) = \sum_{n=0} a_n\, x^n$$

we find

$$a_{n+2} = \frac{n-1}{n+1}\, a_n, \quad n \geq 0$$

solving this recursion relation we get

$$a_{2n} = -\frac{a_0}{2n+1}, a_3 = 0, \quad a_{2n+1} = 0, n \geq 0$$

Hence the solution is of the form

$$y(x) = a_1\, x + a_0 [1 - \sum_{n=1} \frac{x^{2n}}{2n-1}]$$

Initial conditions choose the constants a_0 and a_1 as $a_0 = a_1 = 1$

[25]**4**. **(a)** Let y be a solution of

$$ay'' + by' + cy = 0$$

where a, b, c are real numbers. Show that the number of zeros of $y(x)$ on $(-\infty, \infty)$ is either 0, 1, or ∞
(b) If a, b, c are all postive, show that $y \to 0$ as $x \to \infty$

solution: **(a)**: We have three classes of solutions: $y_1(x) = Ae^{r_1 x} + Be^{r_2}$ (different real roots), $y_2(x) = e^{r_1 x}\,[A + B\,x]$ (equal real roots), and $y_3(x) = e^{-(b/2a)x}\,[A\cos(\alpha x) + B\sin(\alpha x)]$ (complex roots), where $\alpha = \sqrt{\Delta}$. y_1 and y_2 may have zero or one roots depending on the coefficients A and B. The third function may have infinite number zeros due to the trigonometric functions $\sin\alpha x$ and $\cos\alpha x$

(b) Since

$$r_1 = \frac{1}{2a}[-b + \sqrt{b^2 - 4ac}] < 0$$
$$r_2 = \frac{1}{2a}[-b - \sqrt{b^2 - 4ac}] < 0$$

and the exponential terms in the above classes (part **a**) have negative exponents hence all go to zero as $x \to \infty$

[25]**5**. Find the general solution of

$$y''' - 2y'' + y' = x^2 + 2e^x$$

solution:

$$y(x)\frac{1}{3}x^3 + 2x^2 + 6x + x^2e^x + C_1 + e^x\,(C_2 + C_3x)$$