MATH 225: DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA: Final Exam: May 24, 2006

Final Exam Solutions

1.a) Let A be an $n \times n$ matrix and k is a (constant) scalar. Show that the set of all vectors $\vec{v}$ such that $A\,\vec{v} = k\vec{v}$ is a subspace of $\mathbb{R}^n$. (This means that eigenspace associated with an eigenvalue is a subspace of $\mathbb{R}^n$).

Solution: Let W be the subset of $\mathbb{R}^n$ such that any vector $\vec{v}$ in W is an eigenvector of A associated with the eigenvalue k,i.e., $A\,\vec{v} = k\vec{v}$. W is a subspace of $\mathbb{R}^n$ because (**i**) Let $\vec{v}_1$ and $\vec{v}_2$ be in W the their sum $\vec{v}_1 + \vec{v}_2$ is also in W.

$$A\,(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = k\vec{v}_1 + k\vec{v}_2 = k(\vec{v}_1 + \vec{v}_2)$$

(ii) a scalar multiple of $\vec{v}$ $(c\vec{v})$ is also in W, i.e., $A(c\vec{v}) = cA\vec{v} = ck\vec{v} = k(c\vec{v})$

1.b) Prove that the similar matrices have the same eigenvalues.

Solution: Let A and B be $n \times n$ similar matrices, i.e., there exists an invertible matrix P such that $B = PAP^{-1}$. Let the characteristic polynomial of A is given by $\det(A - \lambda I) = f(\lambda)$. Eigenvalues of A are the zeros of $f(\lambda) = 0$. Since $A = P^{-1}BP$ then characteristic polynomial of B can be found as

$$\det(A-\lambda I) = \det(P^{-1}BP-\lambda I) = \det[P^{-1}(B-\lambda I)P] = \det(B-\lambda I) = f(\lambda)$$

Then B has the same characteristic polynomial. Hence A and B have the same eigenvalues.

2.a) Using the Eigenvalue Method for Linear Systems find the solution of the following initial value problem

$$\begin{aligned} \dot{x_1} &= x_1 + 9\,x_2, \\ \dot{x_2} &= -x_1 - 5\,x_2 \end{aligned}$$

with $x_1(0) = 1$ and $x_2(0) = -1$.

Solution: One finds repeated eigenvalue of A $(\lambda = -2)$

$$\begin{aligned} x_1(t) &= 3e^{-2t} - 2(3t+1)e^{-2t} \\ x_2(t) &= -e^{-2t} + 2te^{-2t} \end{aligned} \tag{1}$$

2.b) Using the Eigenvalue Method for Linear Systems find the solution of the following initial value problem

$$\begin{aligned} \dot{x_1} &= -\frac{1}{2}\, x_1 + x_2, \\ \dot{x_2} &= -x_1 - \frac{1}{2}\, x_2 \end{aligned}$$

with $x_1(0) = 1$ and $x_2(0) = 2$.

Solution: One finds a complex eigenvalue of A ($\lambda = -1/2 + i$).

$$\begin{aligned} x_1(t) &= e^{-t/2}\cos t + 2e^{-t/2}\sin t \\ x_2(t) &= -e^{-t/2}\sin t + 2e^{-t/2}\cos t \end{aligned}$$

3.a) Find the general solution of following linear of differential equation

$$y'' + 9y = 2\sec 3x$$

Solution:

$$y_c = A\sin 3x + B\cos 3x$$

where A and B are arbitrary constants. Hence $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$. The using the variation of parameters we have

$$y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x) = C_1(x)\sin 3x + C_2(x)\cos 3x$$

Letting

$$C_1'\sin 3x + C_2'\cos 3x = 0$$

and using y_p in the equation we get

$$3C_1'\cos 3x - 3C_2'\sin 3x = 2\sec 3x$$

We the solve C_1 and C_2 as

$$C_1' = 2/3, \qquad C_2' = -\frac{2}{3}\frac{\sin 3x}{\cos 3x}$$

Then the general solution is given by

$$y(x) = y_c + y_p = A\sin 3x + B\cos 3x + \frac{2}{3}\,x\sin 3x + \frac{2}{9}\ln|\cos 3x|\cos 3x$$

3.b) Diagonalize the matrix A given by

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$$

Solution: Page 375 Example 3. Eigenvalues and their eigenvectors are found as

$$\begin{aligned} \lambda_1 &= 3: \quad \vec{v}_1 = (1,0,2), \\ \lambda_2 &= 1: \quad \vec{v}_2 = (0,2,-5), \\ \lambda_3 &= 0: \quad \vec{v}_3 = (0,1,-3), \end{aligned}$$

Hence the eigenvector matrix is $A = PDP^{-1}$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & -5 & -3 \end{bmatrix}$$

which leads to the diagonal matrix

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4.a). Find the general solution of the following differential equation

$$y'''' + y''' - \mathbf{3}\,y'' - 5y' - 2y = 0$$

Solution: One can write the characteristic equation

$$r^4 + r^3 - 3r^2 - 5r - 2 = (r+1)^3(r-2)$$

Hence we have $r = -1$ with multiplicity 3 and $r = 2$. Then the solution is simply given by

$$y(x) = (A + Bx + Cx^2)e^{-x} + De^{2x}$$

where A, B, C, D are arbitrary constants

4.b) Find the general solution of the following differential equation

$$y''' - 4y' = x + 3\cos x + e^{-2x}$$

Solution: Complementary solution is found easily as

$$y_c(x) = C_1 + C_2e^{2x} + C_3e^{-2x}$$

where C_1, C_2, C_3 are arbitrary constants. In $f(x)$ we have the terms x and e^{-2x} whose derivatives are contained in y_c hence

$$y_p(x) = (A + Bx)x + C\cos x + D\sin x + Exe^{-2x}$$

Inserting this into the differential equation we get

$$A = 0,\ B = -1/8,\ C = 0,\ D = -3/5,\ E = 1/8$$

Hence the general solution is

$$y(x) = C_1 + C_2e^{2x} + C_3e^{-2tx} - \frac{1}{8}x^2 - \frac{3}{5}\sin x + \frac{1}{8}txe^{-2x}$$

5.a) Suppose that the sequence $\{\vec{x}_0, \vec{x}_1, \cdots, \vec{x}_n, \cdots\}$ vectors, where $\vec{x}_k = (a_k, b_k)$,is defined by its initial vector $\vec{x}_0 = (a_0, b_0)$ and the transition matrix A in the following way $\vec{x}_{k+1} = A\,\vec{x}_k$ for all $k = 0, 1, 2, \cdots$. Find a_k and b_k in terms of the initial values a_0 and b_0 when the transition matrix is given by

$$A = \begin{bmatrix} 0.4 & 0.3 \\ -0.4 & 1.2 \end{bmatrix},$$

and $2a_0 - 3b_0 = 1$. Give an approximate solution for a_k and b_k when k is large.

Solution: (Page 235 Example 4) One can show that

$$\vec{x}_k = A^k \, \vec{x}_0, \quad k = 0, 1, 2, \cdots$$

Hence we have to find the kth power A. For this purpose we should diagonalize it. Let P be the matrix diagolizing the matrix A then

$$A = PDP^{-1}$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where λ_1 and λ_2 are the eigenvalues of A. For A we have $\lambda_1 = 1$ and $\lambda_2 = 3/5 = 0.6$ and

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \quad P^{-1} = -\frac{1}{4} \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix},$$

then

$$A^k = PD^kP^{-1},$$

For large k, $(0.6)^k$ is closer to zero hence we find

$$\vec{x}_k = -\frac{1}{4} \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

Hence $a_k = -\frac{1}{4}$ and $b_k = -\frac{1}{2}$ for large k.

5.b) Solve the following differential equation

$$(6xy - y^3)\, dx + (4y + 3x^2 - 3xy^2)\, dy = 0$$

Solution: Page 68 Example 9

$$M(x, y) = 6xy - y^3, \quad N(x, y) = 4y + 3x^2 - 3xy^2$$

one can show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6x - 3y^2$$

hence the equation is exact. Solution can be found from $F_x = M = 6xy - y^3$ which leads to

$$F(x,y) = 3x^2y - xy^3 + f(y)$$

We find $f(y)$ from $F_y = N = 4y + 3x^2 - 3xy^2$ which leads to $f(y) = 2y^2$ hence the solution is

$$F(x,y) = 3x^2y - xy^3 + 2y^2 = C = \text{constant}$$