MATH 225: DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

Second Midterm Exam Solutions

1. Prove the following statements: **1.a)** Let A be an $n \times n$ invertible matrix. Then the homogeneous linear system $A\vec{x} = 0$ has only the trivial solution. *Solution: Since A has inverse, multiply the equation $A\,\vec{v} = 0$ by A^{-1} from the left one gets $\vec{v} = 0$.* **1.b)** A set of vectors in a vector space containing the zero vector can not be linearly independent. *Solution: Let $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k, \vec{0}\}$ be a set of vectors including the zero vector, then*

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k + c_{k+1}\,\vec{0} = \vec{0}$$

is possible for $c_1, c_2, \cdots, c_k, c_{k+1}$ not all vanishing. For instance $c_1 = c_2 = \cdots = c_k = 0$ but $c_{k+1} \neq 0$. **1.c)** Let $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k\}$ be a set of orthogonal vectors in $\mathbb{R}^n$ $(k \leq n)$. Then they are linearly independent. *Solution: We assume that all $\vec{v}_i \neq \vec{0}$, $(i = 1, 2, \cdots, k)$, otherwise from (1.b) they cannot be linearly independent. Let us assume that*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

Then first take the dot product of both sides of this equation by $\vec{v}_1$ and using the orthogonality of $\vec{v}_1$ with others we get $c_1 = 0$. Then repeating this process with all the other vectors we find $c_2 = c_2 = \cdots = c_k = 0$.

2.a) A subset V of $\mathbb{R}^4$ is defined by means of the condition $w = x + y + z$ on a typical vector (x, y, z, w). Prove that V is a subspace of $\mathbb{R}^4$ and find a basis of V. *Solution: Let $\vec{v}_1 = (x_1, y_1, z_1, w_1)$ and $\vec{v}_2 = (x_2, y_2, z_2, w_2)$ be any two vectors in V. This means that $w_1 = x_1 + y_1 + z_1$, and $w_2 = x_2 + y_2 + z_2$. Then* **i)** *$\vec{v}_1 + \vec{v}_2 = (x_1 + y_2, y_1 + y_2, z_1 + z_2, w_1 + w_2)$ such that*

$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (w_1 + w_2)$$

Hence $\vec{v}_1 + \vec{v}_2 \in V$. **ii)** *Let c be a scalar, the $c\vec{v} = (cx, cy, cz, cw)$ such that $cw = cx + cy + cz$ hence $c\vec{v}$ is also in V. Then V is a subspace of $\mathbb{R}^4$. A basis of V: An arbitrary vector $\vec{v} = (x, y, z, w) = (x, y, z, x + y + z)$. Hence*

$$\vec{v} = x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3$$

where $\vec{v}_1 = (1, 0, 0, 1)$, $\vec{v}_2 = (0, 1, 0, 1)$, $\vec{v}_3 = (0, 0, 1, 1)$ form a basis of V.

2.b) Let $P_3(x)$ be set of all polynomials of degree 3 on an interval I. Prove that $P_3(x)$ is a vector space of dimension 4. *Solution: Let $p_1 = a_0 + a_1x + a_2x^2 + a_3x^3$ and $p_2 = b_0 + b_1x + b_2x^2 + b_3x^3$ be two polynomials in P_3 where $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ are real constants. Then it is clear that the sum $p_1 + p_2$ is also in P_3 and any scalar multiple of a polynomial of degree 3 is a polynomial of degree 3, i.e., $cp_1 = (ca_0) + (ca_1)x + (ca_2)x^2 + (ca_3)x^3$. Hence P_3 is a vector space. A possible basis of P_3 is $\{1, x, x^2, x^3\}$ hence dimension of P_3 is four.*

3a.) Let V be a subspace of $\mathbb{R}^5$ spanned by the vectors $\vec{v}_1 = (2, 5, 5, 4, 3)$ and $\vec{v}_2 = (3, 7, 8, 8, 8)$. Find a basis for the orthogonal compliment V^+ of V. *Solution: A basis of V^+ is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ where*

$$\vec{u}_1 = (-5, 1, 1, 0, 0),\ \vec{u}_2 = (-12, 4, 0, 1, 0),\ \vec{u}_3 = (-19, 7, 0, 0, 1)$$

3b.) Solve the initial value problem

$$y'' - 6y' + 25y = 0$$

with $y(0) = 0$ and $y'(0) = 1$. *Solution:* $y(x) = \frac{1}{4}\, e^{3x} \sin 4x$

4.a) Solve the following linear system by the use of Cramer's rule

$$\begin{aligned} 3x_1 + 4x_2 - 3x_3 &= 5, \\ 3x_1 - 2x_2 + 4x_3 &= 7, \\ 3x_1 + 2x_2 - x_3 &= 3, \end{aligned}$$

Solution: $x_1 = -\frac{7}{3},\ x_2 = 9,\ x_3 = 8$

4.b). Let

$$A = \begin{bmatrix} 1 & 2 & 1 & -3 & -3 \\ 2 & 5 & 6 & -10 & -12 \end{bmatrix}$$

Its reduced echelon matrix is

$$A = \begin{bmatrix} 1 & 0 & -7 & 5 & 9 \\ 0 & 1 & 4 & -4 & -6 \end{bmatrix}$$

Hence Row(A) is spanned by the vectors $\{\vec{r}_1, \vec{r}_2\}$ where $\vec{r}_1 = (1, 2, 1, -3, -3)$ and $\vec{r}_2 = (2, 5, 6, -10, -12)$. It has dimension 2. Col(A) is spanned by the vectors $\{\vec{c}_1, \vec{c}_2\}$ where $\vec{c}_1 = (1, 0)$ and $\vec{c}_2 = (0, 1)$. It has dimension 2. To find the Null(A) we write the equivalent reduced system and find $x_3 = r, x_4 = s, x_5 = t$ and

$$x_1 = 7r - 5s - 9t, \quad x_2 = -4r + 4s + 6t$$

then any vector in Null(A) has the form

$$r\vec{u}_1 + s\vec{u}_2 + t\vec{u}_3$$

where $\vec{u}_1 = (7, -4, 1, 0, 0)$, $\vec{u}_2 = (-5, 4, 0, 1, 0)$, $\vec{u}_3 = (-9, 6, 0, 0, 1)$. Hence Null(A) is spanned by these vectors and the dimension of Null(A) is 3. Rank of A is 2.