MATH 225: DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

Solutions of First Midterm Exam
March 13, 2006

1.a) Show that the function $y(x) = A + Be^x + Ce^{2x}$ satisfies the differential equation $y''' - 3y'' + 2y' = 0$, where A, B, and C are arbitrary constants.

Solution:

$$\begin{aligned} y(x) &= A + Be^x + Ce^{2x}, \\ y'(x) &= Be^x + 2Ce^{2x}, \\ y''(x) &= Be^x + 4Ce^{2x} \\ y'''(x) &= Be^x + 8Ce^{2x} \end{aligned}$$

Hence

$$y''' - 3y'' + 2y' = Be^x + 8Ce^{2x} - 3(Be^x + 4Ce^{2x}) + 2(Be^x + 2Ce^{2x}) = 0$$

1.b) Find the constants A, B, and C so that $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$.

Solution:

$$\begin{aligned} y(0) &= A + B + C = 1, \\ y'(0) &= B + 2C = -1, \\ y''(0) &= B + 4C = 0 \end{aligned}$$

There is unique solution $A = 5/2$, $B = -2$, $C = 1/2$.

2.a) Solve the following system by Gauss-Jordan Elimination

$$\begin{aligned} x_1 - 2x_2 + x_3 - 4x_4 &= 1, \\ x_1 + 3x_2 + 7x_3 + 2x_4 &= 2, \\ x_1 - 12x_2 - 11x_3 - 16x_4 &= -1 \end{aligned}$$

Solution :

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 1 & -12 & -11 & -16 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 17/5 & -8/5 & 7/5 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the there are infinitely many solutions with two free parameters

$$\begin{aligned} x_1 &= \frac{7}{5} - \frac{17t}{5} + \frac{8s}{5}, \\ x_2 &= \frac{1}{5} - \frac{6t}{5} - \frac{6s}{5}, \\ x_3 &= t, \quad x_4 = s \end{aligned}$$

2.b) What conditions must b_1,b_2, and b_3 satisfy in order for the system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1, \\ x_1 \quad\quad + x_3 &= b_2, \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

to be consistent. Find the solutions.

Solution :

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_1 - b_3 + b_2 \end{bmatrix}$$

Hence for consistency $b_3 = b_1 + b_2$. In this case there are infinitely many solutions $x_1 = b_2 - t$, $x_2 = b_1 - b_2 + t$, $x_3 = t$ where $t \in \mathbb{R}$ is an arbitrary real variable.

3.a) Solve the differential equation $x^2y' + 2xy = 5y^4$ with $y(1) = 1$. Discus the existence and uniqueness of the solution of the differential equation with the initial condition $y(0) = 0$.

Solution : Let $v(x) = y^{-3}$ then

$$v' - \frac{6}{x}\, v + \frac{15}{x^2} = 0$$

This is a linear equation with the integrating factor $\rho(x) = \frac{1}{x^6}$. Hence the solution is

$$v(x) = C\, x^6 + \frac{15}{7}\, \frac{1}{x} \quad \rightarrow y(x) = \left[\frac{5x}{7Cx^7 + 15} \right]^{1/3}$$

where C is an arbitrary constant. If $y(1) = 1$ then $C = -\frac{8}{7}$.

The singular solution of this equation is $y = 0$. Furthermore at $x = 0$ the existence and uniqueness is not guaranteed. It is clear from the solution that $y(0) = 0$ is satisfied for all C, hence there are infinitely many solutions.

3.b) Solve the differential equation $(x + \ln y)dx + (\frac{x}{y} + e^y)dy = 0$ with $y(0) = 1$

Solution: $M(x, y) = x + \ln y$ and $N(x, y) = \frac{x}{y} + e^y$. Then

$$F_x = M(x, y) = x + \ln y \quad \rightarrow F(x, y) = \frac{x^2}{2} + x \ln y + g(y),$$

but

$$F_y = \frac{x}{y} + g' = \frac{x}{y} + e^y, \quad \rightarrow F(x, y) = \frac{x^2}{2} + x \ln y + e^y$$

Hence the solution is

$$\frac{x^2}{2} + x\ln y + e^y = C$$

where C is an arbitrary constant. Since $y(0) = 1$ then $C = e$. Therefore the solution is

$$\frac{x^2}{2} + x\ln y + e^y = e$$

4.a) Find the Reduced Row-Echelon matrix of the augmented coefficient matrix of the following system. Then find the solution of the system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4, \\ 2x_1 + 5x_2 + 3x_3 &= 5, \\ x_1 \quad\quad + 8x_3 &= 9 \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 3 & 5 \\ 1 & 0 & 8 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & -3 \\ 1 & 0 & 8 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & -2 & 5 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 9 & 10 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence the solution is $x_1 = 1,\ x_2 = 0,\ x_3 = 1$.

4.b) Let $XA = Z$ where A, Z are matrices given by

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 2 & -3 \\ 2 & 0 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 4 & -2 & 3 \\ -1 & 2 & 0 \end{bmatrix}.$$

Find A^{-1} and the matrix X.

Solution:

$$\begin{bmatrix} 0 & -2 & 3 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 & 1 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & -2 & 3 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & -2 & 3 & 1 & 0 & 0 \\ 0 & -4 & 10 & 0 & -2 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & -3/2 & -1/2 & 0 & 0 \\ 0 & -4 & 10 & 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & -3/2 & -1/2 & 0 & 0 \\ 0 & 0 & 4 & -2 & -2 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -5/4 & -3/4 & 3/8 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1/4 \end{bmatrix}$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -5/4 & -3/4 & 3/8 \\ -1/2 & -1/2 & 1/4 \end{bmatrix}, \quad \text{then} \quad X = ZA^{-1} = \begin{bmatrix} 5 & 4 & 0 \\ -7/2 & -5/2 & 3/4 \end{bmatrix}.$$