MATH116: QUIZ IX

Name:
Student Number:
Section: 2
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Question: Find the circulation of the vector field $\mathbf{F} = (x^2 - yz)\mathbf{i} + (y^2 - xz)\mathbf{j} + (z^2 - xy)\mathbf{k}$ around the triangle with vertices $(0,0,1), (1,1,0), (0,0,0)$

Solution:

$$circ(\mathbf{F}) = \oint_C \mathbf{F} \cdot \mathbf{v}\, dt$$

where $C = C_1 \cup C_2 \cup C_3$.

i) C_1 is the line connecting the point $(0,0,0)$ to the point $(1,1,0)$. Hence $\vec{\mathbf{r}} = t\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$. Then $\vec{\mathbf{v}} = \mathbf{i} + \mathbf{j}$ and $\mathbf{F} = t^2\mathbf{i} + t^2\mathbf{j} - t^2\mathbf{k}$ on C_1. Then $\mathbf{F} \cdot \mathbf{v} = 2t^2$ and

$$\int_{C_1} \mathbf{F} \cdot \mathbf{v}\, dt = 2\int_0^1 t^2 dt = 2/3$$

ii) C_2 is the line connecting the point $(1,1,0)$ to the point $(0,0,1)$. Hence $\vec{\mathbf{r}} = (1-t)\mathbf{i} + (1-t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$. Then $\vec{\mathbf{v}} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{F} = (1-3t+2t^2)\,\mathbf{i} + (1-3t+2t^2)\,\mathbf{j} + (2t-1)\,\mathbf{k}$ on C_2. Then $\mathbf{F} \cdot \mathbf{v} = -3+8t-4t^2$ and

$$\int_{C_2} \mathbf{F} \cdot \mathbf{v}\, dt = \int_0^1 (-3 + 8t - 4t^2) dt = -1/3$$

iii) C_3 is the line connecting the point $(0,0,1)$ to the point $(0,0,0)$. Hence $\vec{\mathbf{r}} = -t\mathbf{k}$, $0 \le t \le 1$. Then $\vec{\mathbf{v}} = -\mathbf{k}$ and $\mathbf{F} = t^2\mathbf{k}$ on C_3. Then $\mathbf{F} \cdot \mathbf{v} = -t^2$ and

$$\int_{C_3} \mathbf{F} \cdot \mathbf{v}\, dt = -\int_0^1 t^2 dt = -1/3$$

iv) Circulation of $\mathbf{F}$ around the path C is

$$\oint_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_{C_1} \mathbf{F} \cdot \mathbf{v}\, dt + \int_{C_2} \mathbf{F} \cdot \mathbf{v}\, dt + \int_{C_3} \mathbf{F} \cdot \mathbf{v}\, dt = \frac{2}{3} - \frac{1}{3} - \frac{1}{3} = 0$$