Solution to MATH 116: INTERMEDIATE CALCULUS III
First Midterm Exam, June 18, 2006

Problem 1.a) Prove that

$$\lim_{(x,y)\to(0,0)}(x+y)\sin(x/y)=0$$

Solution 1 (using the $\delta-\varepsilon$ definition) : For a given $\varepsilon>0$ take $\delta=\dfrac{\varepsilon}{2}$. Then for any (x,y) satisfying $0<\sqrt{x^2+y^2}<\delta$, $y\neq 0$, we have

$$\left|(x+y)\sin\frac{x}{y}-0\right|\leq|x+y|\leq|x|+|y|=\sqrt{x^2}+\sqrt{y^2}\leq\sqrt{x^2+y^2}+\sqrt{x^2+y^2}<2\delta=\varepsilon$$

It proves that $\lim\limits_{(x,y)\to(0,0)}(x+y)\sin(x/y)=0$.

Solution 2 (using the Sandwich Theorem) : Since

$$-|x+y|\leq(x+y)\sin\frac{x}{y}\leq|x+y|$$

and $\lim\limits_{(x,y)\to(0,0)}|x+y|=0$ then, by the Sandwich Theorem, $\lim\limits_{(x,y)\to(0,0)}(x+y)\sin\frac{x}{y}=0$.

Problem 1.b) Find the limit, or show that it does not exist.

$$\lim_{(x,y)\to(0,0)}\frac{xy^2}{yx^2+y^4}$$

Solution: Since

$$\lim_{(x,y)\to(0,0),y=x}\frac{xy^2}{yx^2+y^4}=\lim_{x\to0}\frac{x^3}{x^3+x^4}=1$$

and

$$\lim_{(x,y)\to(0,0),y=x^2}\frac{xy^2}{yx^2+y^4}=\lim_{x\to0}\frac{x^5}{x^4+x^8}=0$$

then by the the Two-Path test for nonexistence of a limit, the limit

$$\lim_{(x,y)\to(0,0)}\frac{xy^2}{yx^2+y^4}$$

does not exist.

Problem 2.a) Let $z = yf(x^2 - y^2)$, where f is a differentiable function. Show that

$$y^2 z_x + xyz_y = xz$$

Solution: We are given that $f(t)$ is a differentiable function and $t = x^2 - y^2$. By the Chain Rule,

$$z_x = 2xyf_t \qquad \text{and} \qquad z_y = f - 2y^2 f_t\,.$$

Therefore,

$$y^2 z_x + xyz_y = 2xy^3 f_t + xyf - 2xy^3 f_t = xyf = xz\,.$$

Problem 2.b) Find first partial derivatives of z defined through the equation

$$e^{z-1}x^2 + xy^2 z = 2$$

at the point $(1, 1, 1)$.

Solution: We differentiate both sides of the equation with respect to x, holding y fixed, and treating z as a differentiable function of x:

$$z_x e^{z-1}x^2 + 2xe^{z-1} + y^2 z + xy^2 z_x = 0,$$

$$z_x = -\frac{2xe^{z-1} + y^2 z}{e^{z-1}x^2 + xy^2} \quad \Rightarrow \quad z_x(1,1) = -\frac{2+1}{1+1} = -\frac{3}{2}$$

Similarly,

$$z_y e^{z-1}x^2 + 2yxz + xy^2 z_y = 0,$$

$$z_y = -\frac{2yxz}{e^{z-1}x^2 + xy^2} \quad \Rightarrow \quad z_y(1,1) = -\frac{2}{1+1} = -1$$

Problem 3.a) Let $f(x,y) = xy + y\cos x$. Find the linear approximation of f at the origin.

Solution: Since $f(0,0) = 0$,

$$f_x(x,y) = y - y\sin x \qquad \rightrightarrows \qquad f_x(0,0) = 0\,,$$

$$f_y(x,y) = x + \cos x \qquad \rightrightarrows \qquad f_y(0,0) = 1$$

then the linearization $L(x,y)$ of function $f(x,y)$ at point $(0,0)$ is

$$L(x,y) = y$$

Problem 3.b) (*continuation of the previous problem 3.a*) Let R be region in $\mathbb{R}^2$ given by

$$R : |x| \leq 0.1 \quad |y| \leq 0.1$$

Find an upper bound for the magnitude of the error in the linear approximation of $f(x,y)$, in R.

Solution: We have,

$$f_{xx}(x,y) = -y\cos x, \quad f_{yy} = 0, \quad f_{xy} = 1 - \sin x$$

Therefore, for any point (x,y) belonging to R,

$$|f_{xx}(x,y)| \leq |y| \leq 0.1, \quad |f_{yy}(x,y)| = 0, \quad |f_{xy}(x,y)| \leq 1 + |\sin x| \leq 2$$

. It implies that $M \leq 2$ and

$$|E(x,y)| \leq \frac{1}{2}M(|x| + |y|)^2 \leq \frac{1}{2}\cdot 2(0.1 + 0.1)^2 = 0.04$$

Problem 4.a). Let $S_1 : z = x^2 - y^2$ and $S_2 : xyz + 30 = 0$ be two surfaces in $\mathbb{R}^3$. Find the equation of the tangent line of the intersection curve of the these two surfaces at the point $(-3, 2, 5)$.

Solution: Notice that surface S_1 is a level surface of function $f(x, y, z) = z - x^2 + y^2$ and surface S_2 is a level surface of function $g(x, y, z) = xyz + 30$. Since the intersection curve belongs to both level surfaces then the direction of tangent line to this curve at point $(-3, 2, 5)$ is parallel to the vector $\nabla f \times \nabla g|_{(-3,2,5)}$. We have,

$$\nabla f = -2x\vec{i} + 2y\vec{j} + \vec{k}, \quad \nabla f|_{(-3,2,5)} = 6\vec{i} + 4\vec{j} + \vec{k}$$

$$\nabla g = yz\vec{i} + xz\vec{j} + xy\vec{k}, \quad \nabla g|_{(-3,2,5)} = 10\vec{i} - 15\vec{j} - 6\vec{k}$$

$$\nabla f \times \nabla g|_{(-3,2,5)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 4 & 1 \\ 10 & -15 & -6 \end{vmatrix} = -9\vec{i} + 46\vec{j} - 130\vec{k}$$

Therefore, the tangent line to the intersection curve passing through the point $(-3, 2, 5)$ is

$$\begin{array}{rrll} x = & -3 & -9t, & \\ y = & 2 & +46t, & -\infty < t < \infty \\ z = & 5 & -130t & \end{array}$$

Problem 4.b) Let f be a function of two variables x and y and P a point in $\mathbb{R}^2$. Derivatives of f in the directions $\vec{u} = \frac{1}{\sqrt{5}}(1, 2)$ and $\vec{v} = \frac{1}{5}(-3, 4)$ are given as

$$(D_u f)_P = \sqrt{5}, \quad (D_v f)_P = 1$$

Find the derivative of f in the direction of $\vec{w} = \frac{1}{\sqrt{2}}(1, -1)$ at P.

Solution: We have,

$$\begin{cases} (D_u f)_P = \sqrt{5} \\ (D_v f)_P = 1 \end{cases} \rightrightarrows \begin{cases} \frac{\partial f}{\partial x}|_P \frac{1}{\sqrt{5}} + \frac{\partial f}{\partial y}|_P \frac{2}{\sqrt{5}} = \sqrt{5} \\ \\ \frac{\partial f}{\partial x}|_P \frac{-3}{5} + \frac{\partial f}{\partial y}|_P \frac{4}{5} = 1 \end{cases} \rightrightarrows \begin{cases} \frac{\partial f}{\partial x}|_P = 1 \\ \\ \frac{\partial f}{\partial y}|_P = 2 \end{cases}$$

Therefore, $\nabla f|_P = \vec{i} + 2\vec{j}$, and hence,

$$(D_w f)_P = \nabla f|_P \circ w = (\vec{i} + 2\vec{j}) \circ (\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}) = \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

Problem 5) Find the absolute maximum and absolute minimum values of the function $f(x,y) = x^2 - 3y^2 + 2xy + x + 3$ on the triangular region R bounded by the lines $x = 0$, $y = 0$, $y + x + 5 = 0$. Find also the saddle points (if any)

Solution: Since f is differentiable, the only places where f can assume absolute maximum and absolute minimum values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary of R.
Interior points: For these we have

$$\begin{cases} f_x = 2x + 2y + 1 = 0 \\ f_y = -6y + 2x = 0 \end{cases} \Longrightarrow \begin{cases} x = -3/8 \\ y = -1/8 \end{cases}$$

$(-3/8, -1/8)$ is an interior point of R. To check whether it is a saddle point, let us calculate the discriminant $f_{xx}f_{yy} - f_{xy}^2$ of f at this point. We have $f_{xx} = 2$, $f_{yy} = -6$, $f_{xy} = 2$ and, therefore, $f_{xx}f_{yy} - f_{xy}^2 = -12 - 4 = -16 < 0$, that implies that $(-3/8, -1/8)$ is a saddle point of function f.
Boundary points: The boundary of triangle R consists of three line segments:
$A_1 : y = 0, -5 \leq x \leq 0$; $A_2 : x = 0, -5 \leq y \leq 0$; $A_3 : y = -x-5, -5 \leq x \leq 0$.
On the segment A_1 function $f(x,y) = f(x,0) = x^2 + x + 3$ may have extreme values at the end points $(0,0)$, $(-5,0)$ and at the point where $f'(x,0) = 2x + 1 = 0$, i.e. at $(-1/2, 0)$. We have

$$f(0,0) = 3, \quad f(-5,0) = 23, \quad f(-1/2,0) = 2\frac{3}{4}$$

On the segment A_2 function $f(x,y) = f(0,y) = -3y^2 + 3$ may have extreme values at the end points $(0,0)$, $(0,-5)$ and at the point where $f'(0,y) = -6y = 0$,which coincides with one of the end points. We have

$$f(0,0) = 3, \quad f(0,-5) = -72$$

On the segment A_3 function $f(x,y) = f(x,-x-5) = x^2 - 3(x+5)^2 - 2x(x+5) + x + 3 = -4x^2 - 39x - 72$ may have extreme values at the end points $(0,-5)$, $(-5,0)$(that are already studied) and at the point where $f'(x,-x-5) = -8x - 39 = 0$, i.e. at $(-39/8, -1/8)$. We have

$$f(-39/8, -1/8) = 23\frac{4}{64}$$

Therefore, the absolute maximum and absolute minimum values of f taken on R are $23\frac{4}{64}$ and -72 respectively.