Math 101
Second Midterm Exam
April 30, 2011
12.30 - 14:30

Name : ____________________

ID# : ____________________

Department : ____________________

Section : ____________________

• The exam consists of 5 questions of equal weight.

• Read the questions carefully.

• **Solutions, not answers, get points.** Show all your work in well-organized mathematical sentences. Explain your reasoning carefully and in full.

• **What can not be read will not be read.** Write clearly and cleanly.

• Simplify your answers as far as possible unless otherwise stated.

• Calculators and dictionaries are not allowed.

• Turn off all electronic devices including your cell phones before the exam starts.

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

1. Find the asymptotes, the critical points, maxima and minima, intervals of increasing/decreasing, points of inflection, intervals of concavity of the function $f(x) = (1 + x^3)^{1/3}$. Sketch the graph of f.

SOLUTION: $y = (1 + x^3)^{1/3}$.

1. Oblique asymptote $\quad y = x$

2. $y = 0$ for $x = -1$

3. Critical Points: $y' = x^2\,(1 + x^3)^{-2/3}$. Hence $x = 0$ and $x = -1$ are critical points.

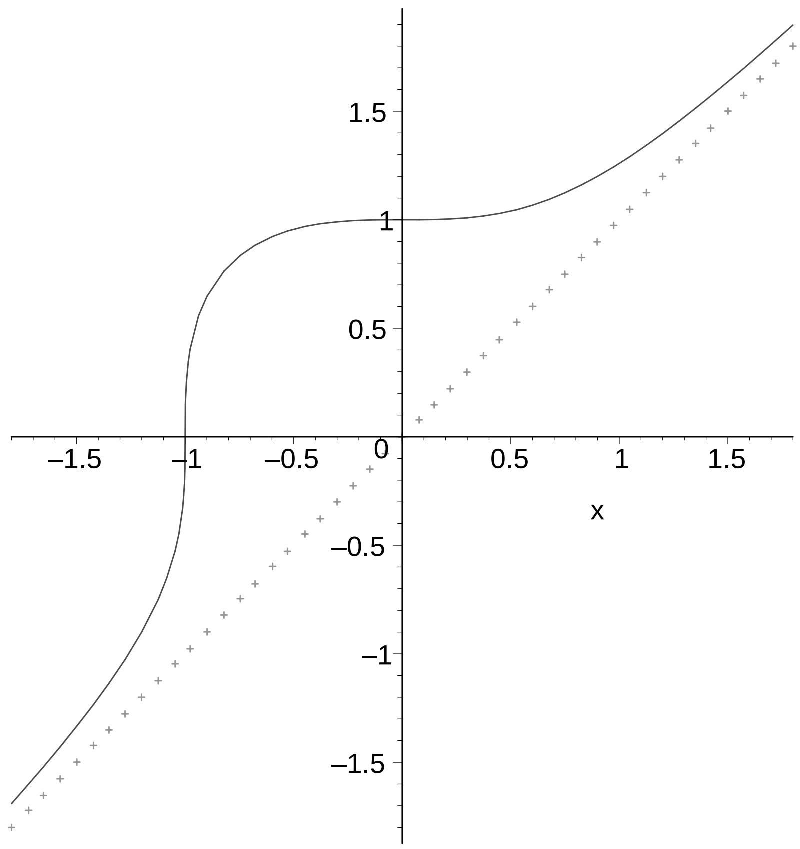
4. $f(x)$ is an increasing function for all x (since y' is a square).

5. Concavity: $y'' = 2\,x\,(1 + x^3)^{-5/3}$. Hence

$y'' > 0$ for $x < -1$ (concave up),
$y'' < 0$ for $-1 < x < 0$ (concave down),
$y'' > 0$ for $x > 0$ (concave up)

5. Inflection points are $x = 0$ and $x = -1$. (Note that, even though $f'(-1) = \infty$, the still has a tangent at this point.)

6. Sketch of the curve:



2a. Find the limit $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$

SOLUTION: ($\overset{*}{=}$ stands for L'Hôpital's rule)

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\left(\frac{\ln x-x+1}{(x-1)\ln x}\right)$$

$$\overset{*}{=}\lim_{x\to 1}\left(\frac{1/x-1}{\ln x+1-1/x}\right)=\lim_{x\to 1}\left(\frac{1-x}{x\ln x+x-1}\right)\overset{*}{=}\lim_{x\to 1}\left(\frac{-1}{\ln x+1+1}\right)=-\frac{1}{2}.$$

2b. Find the limit $\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{n^2+k^2}+\cdots+\frac{n}{n^2+n^2}\right)$. (Explain what you are doing).

SOLUTION: This problem is from the homework set.

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\cdots+\frac{n}{n^2+k^2}+\cdots+\frac{n}{n^2+n^2}\right)=$$

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+1/n^2}+\frac{1}{1+4/n^2}+\cdots+\frac{1}{1+k^2/n^2}+\cdots+\frac{1}{1+1}\right)=$$

From the definition of definite integrals this limit is

$$\int_0^1\frac{dx}{1+x^2}=\tan^{-1}(x)|_0^1=\frac{\pi}{4}.$$

3a. Find all functions $f(x)$ and all real numbers a such that $\int_x^a f(t)\,dt=2-2\sqrt{x}$ for all $x>0$. (Explain what you are doing).

SOLUTION:

a) At $x=a$ we get $0=2-2\sqrt{a}$ which gives $a=1$.
b) Differentiating both sides with respect x we get

$$f(x)=\frac{1}{\sqrt{x}}.$$

3b. Find the area of the region bounded by the curves $x=y^2-4y$ and $x=2y-y^2$ (Explain what you are doing).

SOLUTION: To find the intersection points, solve $y^2-4y=2y-y^2$. The points are $(0,0)$ and $(3,-3)$. For the area, it is much more convenient to integrate with respect to y:

$$A(R)=\int_0^3[2y-y^2-(y^2-4y)]dy=\int_0^3(6y-2y^2)dy=(3y^2-\frac{2}{3}y^3)|_0^3=9.$$

4a. Find the volume of the solid generated by revolving the region bounded by the curves $y = 2x - x^2$ and $y = 0$ about the x-axis (Explain what you are doing).

SOLUTION: Along the x-axis, the region is bounded by 0 and 2. (Solve the equation $2x - x^2 = 0$.) Using the disk method we get the volume V of the solid generated as

$$V = \int \pi\, y^2\, dx = \pi \int_0^2 (2x - x^2)^2\, dx = \pi \int_0^2 (4x^2 - 4x^3 + x^4) dx$$

$$= \left(\frac{4}{3}\, x^3 - x^4 + \frac{1}{5}\, x^5 \right) \Bigg|_0^2 = \frac{16\pi}{15}.$$

One can also solve this problem by using the "cylindrical shell" method". In this case we have

$$V = 2\pi \int_0^1 y\,(2 - 2x)\, dy = 4\pi \int_0^1 y\,(1 - x)\, dy = 4\,\pi \int_0^1 y\, \sqrt{1 - y}\, dy = \frac{16\pi}{15}$$

4b. Find the volume of the solid generated by revolving the region bounded by the curves $y = 2x - x^2$ and $y = 0$ about the y-axis (Explain what you are doing).

SOLUTION: Using the "cylindrical shell" method we get the volume V of the solid generated as

$$V = \int 2\pi x y dx = 2\pi \int_0^2 x(2x - x^2) dx = 2\pi\,(\frac{2}{3}\,x^3 - \frac{1}{4}\,x^4)|_0^2 = 2\pi\,(\frac{16}{3} - 4) = \frac{8\,\pi}{3}$$

One can also solve this problem by using the "washer method". In this case we have

$$V = \pi \int_0^1 [(2 - x)^2 - x^2]\, dy = 4\,\pi \int_0^1 [1 - x]\, dy = 4\,\pi \int_0^1 \sqrt{1 - y}\, dy = \frac{8\,\pi}{3}$$

5a. Find $\displaystyle\int \frac{x^2}{1 - x^4} dx$. (Explain what you are doing).

SOLUTION: Using partial fractions we get

$$\frac{x^2}{1 - x^4} = \frac{1}{2} \left(\frac{1}{1 - x^2} - \frac{1}{1 + x^2} \right)$$

$$\frac{x^2}{1 - x^4} = \frac{1}{4} \left(\frac{1}{1 - x} + \frac{1}{1 + x} \right) - \frac{1}{2}\, \frac{1}{1 + x^2}$$

Hence

$$\int \frac{x^2}{1 - x^4} dx = \frac{1}{4} \int \frac{dx}{1 - x} + \frac{1}{4} \int \frac{dx}{1 + x} - \frac{1}{2} \int \frac{dx}{1 + x^2}$$

$$= \frac{1}{4} \ln \left| \frac{1 + x}{1 - x} \right| - \frac{1}{2}\, \tan^{-1}(x) + C$$

5b. Evaluate $\int_0^2 \sqrt{4+x^2}\,dx$. (Explain what you are doing).

SOLUTION: Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta\,d\theta$. Hence

$$I = \int_0^2 \sqrt{4+x^2}\,dx = 4\int_0^{\pi/4} |\sec\theta| \sec^2\theta\,d\theta.$$

Since $\cos\theta > 0$ for $0 < \theta < \pi/4$, then

$$I = 4\int_0^{\pi/4} \sec^3\theta\,d\theta$$

The last integral can be found by the use of integration by parts (with $u = \sec\theta$ and $dv = \sec^2\theta\,d\theta$)

$$\int \sec^3\theta\,d\theta = \int \sec\theta\ \sec^2\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\ \tan^2\theta d\theta$$

$$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\ \sec\theta d\theta = \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$$

Hence

$$\int \sec^3\theta d\theta = \frac{1}{2}(\sec\theta\ \tan\theta + \ln|\sec\theta + \tan\theta|)$$

Finally we obtain

$$I = 4\int_0^{\pi/4} \sec^3\theta\,d\theta = 2\sqrt{2} + 2\ln(\sqrt{2}+1)$$