

Math 101  
First Midterm Exam  
March 12, 2011  
12.30 - 14:30

Name : \_\_\_\_\_  
ID# : \_\_\_\_\_  
Department : \_\_\_\_\_  
Section : \_\_\_\_\_

- The exam consists of 5 questions of equal weight.
- Read the questions carefully.
- **Solutions, not answers, get points.** Show all your work in well-organized mathematical sentences. Explain your reasoning carefully and in full.
- **What can not be read will not be read.** Write clearly and cleanly.
- Simplify your answers as far as possible unless otherwise stated.
- Calculators and dictionaries are not allowed.
- Turn off all electronic devices including your cell phones before the exam starts.

\_\_\_\_\_ Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

1a. Show that

$$\lim_{x \rightarrow 0^+} \sqrt{x} e^{\cos \frac{\pi}{x}} = 0$$

**Solution:** Since  $-1 \leq \cos(\pi/x) \leq 1$  for all  $x$ , then

$$e^{-1} \leq e^{\cos(\pi/x)} \leq e$$

Hence

$$e^{-1} \sqrt{x} \leq \sqrt{x} e^{\cos \frac{\pi}{x}} \leq e \sqrt{x}$$

Since  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$  then, by **the sandwich theorem**, we get

$$\lim_{x \rightarrow 0^+} \sqrt{x} e^{\cos \frac{\pi}{x}} = 0$$

1b. Find the limit

$$\lim_{x \rightarrow +\infty} [x\sqrt{3x^2 + 4e^{-x} + 1} - x\sqrt{3x^2 + 2e^{-x}}]$$

**Solution:** This is a limit in **indeterminant** form  $\infty(\infty - \infty)$ .

$$\begin{aligned} & \lim_{x \rightarrow +\infty} [x\sqrt{3x^2 + 4e^{-x} + 1} - x\sqrt{3x^2 + 2e^{-x}}] = \\ & \lim_{x \rightarrow +\infty} x [\sqrt{3x^2 + 4e^{-x} + 1} - \sqrt{3x^2 + 2e^{-x}}] \frac{[\sqrt{3x^2 + 4e^{-x} + 1} + \sqrt{3x^2 + 2e^{-x}}]}{[\sqrt{3x^2 + 4e^{-x} + 1} + \sqrt{3x^2 + 2e^{-x}}]} = \\ & \lim_{x \rightarrow +\infty} \frac{x(2e^{-x} + 1)}{[\sqrt{3x^2 + 4e^{-x} + 1} - \sqrt{3x^2 + 2e^{-x}}]} = \lim_{x \rightarrow +\infty} \frac{(2e^{-x} + 1)}{\sqrt{3 + 4\frac{1}{x^2}e^{-x} + \frac{1}{x^2}} + \sqrt{3 + 2\frac{1}{x^2}e^{-x}}} \\ & = \frac{1}{2\sqrt{3}} \end{aligned}$$

2a. The following function is differentiable everywhere

$$f(x) = \begin{cases} \sin(ax) + b, & x > 0 \\ \sin^2(2x) + 2x, & x \geq 0. \end{cases}$$

Find the constants  $a$  and  $b$ .

**Solution:** This function is differentiable everywhere except the origin. If it is differentiable also at  $x = 0$ , the constants  $a$  and  $b$  will not be so free. To determine them first we use the fact that  $f(x)$  must be **continuous** at  $x = 0$ . The left and right limits of  $f$  at the origin must be equal. This gives the condition  $b = 0$ . The derivative (which exists by assumption) of  $f$  at the origin is

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

This limit exists by assumption hence **the right and left limits exist and are equal**. This gives

$$\lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{\sin^2 2h + 2h}{h} = 2$$

$$\lim_{h \rightarrow 0^-} \frac{\sin ah}{h} = a$$

Hence  $a = 2$  and  $b = 0$  and  $f'(0) = 2$ .

**2b.** Find the derivative of the function  $y = 2(\ln x)^{x/2}$ .

**Solution:** We use here **logarithmic differentiation**. Taking the (natural) logarithm of both sides we get

$$\ln y = \frac{x}{2} \ln \ln x + \ln 2$$

Taking the derivative of both sides with respect to  $x$  (implicit differentiation)

$$\frac{y'}{y} = \frac{1}{2} \ln(\ln x) + \frac{1}{2 \ln x}$$

Hence

$$y' = (\ln x)^{x/2} \left[ \ln(\ln x) + \frac{1}{\ln x} \right]$$

**3a.** Find the tangent and the normal lines to the curve  $x^2 + y^2 = y^4 + 1$  at the point  $(1,1)$ .

**Solution:** We use the **implicit differentiation**. Taking the derivative of both sides with respect to  $x$  we get

$$2x + 2yy' = 4y^3 y'$$

At the point  $(1,1)$  we get  $y'(1) = 1$ . Hence the equation of the tangent line is

$$y = x$$

and the equation of the normal line at the same point is

$$y = -x + 2$$

**3b.** Suppose that the differentiable function  $y = f(x)$  has an inverse and that the graph of  $f(x)$  passes through the point  $(1,4)$  with slope 2. Find the slope of the the graph of  $f^{-1}(x)$  at  $(4,1)$ .

**Solution:** In general if  $f(x)$  is a 1 – 1 function over its domain and if  $f^{-1}(x)$  is its **inverse** then the derivative of the inverse is related to the derivative of the function by

$$\frac{df^{-1}}{dx}(x) = \frac{1}{\frac{df}{du}|_{u=f^{-1}(x)}}$$

Since  $f'(1) = 2$  and  $f(1) = 4$  or  $f^{-1}(4) = 1$ , then

$$\frac{df^{-1}}{dx}(4) = \frac{1}{\frac{df}{du}|_{u=f^{-1}(4)}} = \frac{1}{\frac{df}{du}|_{u=1}} = \frac{1}{2}.$$

**4a.** Find the absolute maximum and absolute minimum values of the function

$$f(x) = \frac{x+1}{x^2+2x+2}$$

on the segment  $[-4,1]$ .

**Solution:** We find the **critical points** of  $f$ .

$$\frac{df}{dx} = -\frac{x(x+2)}{(x^2+2x+2)^2} = 0$$

Critical Points are  $x = 0$  and  $x = -2$ . We then find the values of the function at the critical points and at the end points.

$$f(-4) = -\frac{3}{5}, \quad f(-2) = -\frac{1}{2}, \quad f(0) = \frac{1}{2}, \quad f(1) = \frac{2}{5}$$

**Absolute maximum value** of  $f$  is  $\frac{1}{2}$  at  $x = 0$  and **Absolute minimum value** of  $f$  is  $-\frac{1}{2}$  at  $x = -2$ .

**4b.** Show that the function  $f(x) = x^3 + \frac{4}{x^2} + 7$  has exactly one zero in the interval  $(-\infty, 0)$ .

**Solution:** Since  $f(-1) > 0$  and  $f(-3) < 0$  then by the **intermediate value theorem**  $f(x)$  has a real root  $x_1$  in  $[-3, -1]$  (indeed  $f(-2) = 0$ ). If there exists another root at a point  $x_2$  in  $(-\infty, 0)$  then, **by the Rolle theorem**, there must be a point  $c$  in  $(x_1, x_2)$  so that  $f'(c) = 0$ . But this is impossible because  $f'(x) = 3x^2 - \frac{8}{x^3} > 0$  in  $(-\infty, 0)$ .

**5a.** Sand falls from a conveyor belt at the rate of  $10 \text{ m}^3/\text{min}$  onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high?

**Solution:** Let  $h$  be the height and  $r$  be the radius of the cone at any time. Since by assumption

$$h = \frac{3}{8} 2r = \frac{3r}{4}$$

Since sand pile is in conical shape the volume at any time is

$$V = \frac{1}{3} \pi r^2 h = \frac{\pi}{4} r^3$$

In general the change in volume is found as

$$\frac{dV}{dt} = \frac{3\pi}{4} r^2 \frac{dr}{dt}$$

From this relation, since  $\frac{dV}{dt} = 10 \text{ m}^3/\text{min}$  we find that

$$\frac{dr}{dt} = \frac{40}{3\pi r^2}$$

When  $h = 4 \text{ m}$  or  $r = \frac{16}{3} \text{ m}$  we find that  $\frac{dr}{dt} = \frac{15}{32\pi} = 0.1492 \text{ m/min}$  and  $\frac{dh}{dt} = \frac{45}{128\pi} = 0.1119 \text{ m/min}$ .

**5b.** Show that the function  $y = 2 \sin(\ln x)$  satisfies the equation  $x^2 y'' + xy' + y = 0$ .

**Solution:** Just find the first and second derivatives of the function.

$$y' = \frac{2}{x} \cos(\ln x), \quad y'' = -\frac{2}{x^2} \cos(\ln x) - \frac{2}{x^2} \sin(\ln x)$$

when these inserted in the above equation

$$-x^2 \frac{2}{x^2} [\cos(\ln x) + \sin(\ln x)] + x \frac{2}{x} \cos(\ln x) + 2 \sin(\ln x) = 0$$

we get  $0 = 0$ . This means that the function satisfies the given equation.