Math 101
First Midterm Exam
March 12, 2011
12.30 - 14:30

Name : ____________________

ID# : ____________________

Department : ____________________

Section : ____________________

- The exam consists of 5 questions of equal weight.

- Read the questions carefully.

- **Solutions, not answers, get points.** Show all your work in well-organized mathematical sentences. Explain your reasoning carefully and in full.

- **What can not be read will not be read.** Write clearly and cleanly.

- Simplify your answers as far as possible unless otherwise stated.

- Calculators and dictionaries are not allowed.

- Turn off all electronic devices including your cell phones before the exam starts.

Please do <u>not</u> write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

1a. Show that

$$\lim_{x\to 0^+} \sqrt{x}\, e^{\cos\frac{\pi}{x}} = 0$$

Solution: Since $-1 \le \cos(\pi/x) \le 1$ for all x, then

$$e^{-1} \le e^{\cos(\pi/x)} \le e$$

Hence

$$e^{-1}\sqrt{x} \le \sqrt{x}\, e^{\cos\frac{\pi}{x}} \le e\sqrt{x}$$

Since $\lim_{x\to 0^+} \sqrt{x} = 0$ then, by **the sandwich theorem**, we get

$$\lim_{x\to 0^+} \sqrt{x}\, e^{\cos\frac{\pi}{x}} = 0$$

1b. Find the limit

$$\lim_{x\to +\infty} [x\sqrt{3x^2+4e^{-x}+1} - x\sqrt{3x^2+2\,e^{-x}}\,]$$

Solution: This is a limit in **indeterminant** form $\infty(\infty-\infty)$.

$$\lim_{x\to +\infty} [x\sqrt{3x^2+4e^{-x}+1} - x\sqrt{3x^2+2\,e^{-x}}\,] =$$

$$\lim_{x\to +\infty} x\,[\sqrt{3x^2+4e^{-x}+1} - \sqrt{3x^2+2\,e^{-x}}\,]\frac{[\sqrt{3x^2+4e^{-x}+1} + \sqrt{3x^2+2\,e^{-x}}\,]}{[\sqrt{3x^2+4e^{-x}+1} + \sqrt{3x^2+2\,e^{-x}}\,]} =$$

$$\lim_{x\to +\infty} \frac{x\,(2\,e^{-x}+1)}{[\sqrt{3x^2+4e^{-x}+1} - \sqrt{3x^2+2\,e^{-x}}\,]} = \lim_{x\to +\infty} \frac{(2\,e^{-x}+1)}{\sqrt{3+4\frac{1}{x^2}\,e^{-x}+\frac{1}{x^2}} + \sqrt{3+2\,\frac{1}{x^2}\,e^{-x}}}$$

$$= \frac{1}{2\sqrt{3}}$$

2a. The following function is differentiable everywhere

$$f(x) = \begin{cases} \sin(ax)+b, & x>0 \\ \sin^2(2x)+2x, & x\ge 0. \end{cases}$$

Find the constants a and b.
Solution: This function is differentiable everywhere except the origin. If it is differentiable also at $x=0$, the constants a and b will not be so free. To determine them first we use the fact that $f(x)$ must be **continuous** at $x=0$. The left and right limits of f at the origin must be equal. This gives the condition $b=0$. The derivative (which exists by assumption) of f at the origin is

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

This limit exists by assumption hence **the right and left limits exist and are equal**. This gives

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\sin^2 2h + 2h}{h} = 2$$

$$\lim_{h\to 0^-} \frac{\sin ah}{h} = a$$

Hence $a = 2$ and $b = 0$ and $f'(0) = 2$.

2b. Find the derivative of the function $y = 2(\ln x)^{x/2}$.
Solution: We use here **logarithmic differentiation**. Taking the (natural) logarithm of both sides we get

$$\ln y = \frac{x}{2} \ln \ln x + \ln 2$$

Taking the derivative of both sides with respect to x (implicit differentiation)

$$\frac{y'}{y} = \frac{1}{2} \ln(\ln x) + \frac{1}{2 \ln x}$$

Hence

$$y' = (\ln x)^{x/2}\ [\ln(\ln x) + \frac{1}{\ln x}]$$

3a. Find the tangent and the normal lines to the curve $x^2 + y^2 = y^4 + 1$ at the point (1,1).
Solution: We use the **implicit differentiation**. Taking the derivative of both sides with respect to x we get

$$2x + 2yy' = 4y^3\, y'$$

At the point $(1, 1)$ we get $y'(1) = 1$. Hence the equation of the tangent line is

$$y = x$$

and the equation of the normal line at the same point is

$$y = -x + 2$$

3b. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of $f(x)$ passes through the point (1,4) with slope 2. Find the slope of the the graph of $f^{-1}(x)$ at (4,1).
Solution: In general if $f(x)$ is a $1-1$ function over its domain and if $f^{-1}(x)$ is its **inverse** then the derivative of the inverse is related to the derivative of the function by

$$\frac{df^{-1}}{dx}(x) = \frac{1}{\frac{df}{du}|_{u=f^{-1}(x)}}$$

Since $f'(1) = 2$ and $f(1) = 4$ or $f^{-1}(4) = 1$, then

$$\frac{df^{-1}}{dx}(4) = \frac{1}{\frac{df}{du}\big|_{u=f^{-1}(4)}} = \frac{1}{\frac{df}{du}\big|_{u=1}} = \frac{1}{2}.$$

4a. Find the absolute maximum and absolute minimum values of the function

$$f(x) = \frac{x+1}{x^2+2x+2}$$

on the segment [-4,1].
Solution: We find the **critical points** of f.

$$\frac{df}{dx} = -\frac{x(x+2)}{(x^2+2x+2)^2} = 0$$

Critical Points are $x = 0$ and $x = -2$. We then find the values of the function at the critical points and at the end points.

$$f(-4) = -\frac{3}{5},\ f(-2) = -\frac{1}{2},\ f(0) = \frac{1}{2},\ f(1) = \frac{2}{5}$$

Absolute maximum value of f is $\frac{1}{2}$ at $x = 0$ and **Absolute minimum value** of f is $-\frac{1}{2}$ at $x = -2$.

4b. Show that the function $f(x) = x^3 + \frac{4}{x^2} + 7$ has exactly one zero in the interval $(-\infty, 0)$.
Solution: Since $f(-1) > 0$ and $f(-3) < 0$ then by the **intermediate value theorem** $f(x)$ has a real root x_1 in $[-3, -1]$ (indeed $f(-2) = 0$). If there exists another root at a point x_2 in $(-\infty, 0)$ then, **by the Rolle theorem**, there must be a point c in (x_1, x_2) so that $f'(c) = 0$. But this is impossible because $f'(x) = 3x^2 - \frac{8}{x^3} > 0$ in $(-\infty, 0)$.

5a. Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eights of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high?
Solution: Let h be the height and r be the radius of the cone at any time. Since by assumption

$$h = \frac{3}{8}2r = \frac{3r}{4}$$

Since sand pile is in conical shape the volume at any time is

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{4}r^3$$

In general the change in volume is found as

$$\frac{dV}{dt} = \frac{3\pi}{4}r^2\frac{dr}{dt}$$

From this relation, since $\frac{dV}{dt} = 10$ m^3/min we find that

$$\frac{dr}{dt} = \frac{40}{3\pi r^2}$$

When $h = 4$ m or $r = \frac{16}{3}$ m we find that $\frac{dr}{dt} = \frac{15}{32\pi} = 0.1492$ m/min and $\frac{dh}{dt} = \frac{45}{128\pi} = 0.1119$ m/min.

5b. Show that the function $y = 2\sin(\ln x)$ satisfies the equation $x^2 y'' + xy' + y = 0$.
Solution: Just find the first and second derivatives of the function.

$$y' = \frac{2}{x}\cos(\ln x), \;\; y'' = -\frac{2}{x^2}\cos(\ln x) - \frac{2}{x^2}\sin(\ln x)$$

when these inserted in the above equation

$$-x^2\,\frac{2}{x^2}\left[\cos(\ln x) + \sin(\ln x)\right] + x\,\frac{2}{x}\cos(\ln x) + 2\sin(\ln x) = 0$$

we get $0 = 0$. This means that the function satisfies the given equation.