MATH 101, ALL SECTIONS, HOMEWORK #5 (SPRING 2011)

Due to the week starting May 9, at the first hour of the last lecture day that week.

QUESTION 1.

(a) Find the length of the curve

$$y = \ln \cos x, \quad 0 \le x \le \frac{\pi}{4}.$$

SOLUTION: $y' = -\tan x$, hence

$$L = \int_0^{\pi/4} \sqrt{1+y'^2}dx = \int_0^{\pi/4} \sec x dx = \log|\sec x + \tan x|\Big|_0^{\pi/4} = \ln(1+\sqrt{2}).$$

(b) Find the area of the surface generated by rotation of the curve

$$y = \frac{e^x + e^{-x}}{2}, \quad 0 \le x \le \ln 2$$

about x-axis.

SOLUTION. $y' = \frac{1}{2}(e^x - e^{-x})$, $1 + y'^2 = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 = y^2$, and hence

$$\text{Area} = 2\pi \int_0^{\ln 2} y\sqrt{1+y'^2}dx = \frac{\pi}{2}\int_0^{\ln 2}(e^x + e^{-x})^2 dx = \frac{\pi}{16}(15 + 16\ln 2).$$

QUESTION 2. Evaluate the following integrals by an appropriate substitution and/or integration by parts.

(a) $\int x^3 \ln x dx$;

(b) $\int \sin(\ln x)dx$;

(c) $\int x(\ln x)^2 dx$;

SOLUTIONS.

(a) $\int x^3 \ln x dx = \frac{1}{4}\int \ln x dx^4 = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx = \frac{1}{4}x^4\left(\ln x - \frac{1}{4}\right)$. □

(b) Put $x = e^u$ so that $I = \int \sin(\ln x)dx = \int e^u \sin u du$. Applying integration by parts twice we find:

$$\begin{aligned} I &= \int \sin u de^u = e^u \sin u - \int e^u \cos u du = e^u \sin u - \int \cos u de^u \\ &= e^u \sin u - e^u \cos u - \int e^u \sin u du = e^u \sin u - e^u \cos u - I \end{aligned}$$

whence $I = \frac{1}{2}e^u(\sin u - \cos u) + C = \frac{1}{2}x(\sin\ln x - \cos\ln x) + C$. □

(c)

$$\begin{aligned}\int x(\ln x)^2 dx &= \frac{1}{2}\int (\ln x)^2 dx^2 = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x dx = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}\int \ln x dx^2 \\ &= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{2}\int x dx = \frac{x^2}{2}\left[(\ln x)^2 - \ln x + \frac{1}{2}\right] + C. \square\end{aligned}$$

QUESTION 3. Evaluate the following integrals

(a) $\int \cos 3x \sin 2x \cos x dx$;

(b) $\int \sin^2 x \cos^2 x dx$;

(c) $\int x\sqrt{1-\cos x}\, dx$;

(d) Prove the reduction formula $\int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx$;

SOLUTIONS.

(a) $\int \cos 3x \sin 2x \cos x dx = \frac{1}{2}\int (\sin 5x - \sin x) dx = -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C. \square$

(b) $\int \sin^2 x \cos^2 x dx = \frac{1}{4}\int \sin^2 2x dx = \frac{1}{8}\int (1-\cos 4x) dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C. \square$

(c) $\int x\sqrt{1-\cos x}\, dx = \sqrt{2}\int x \sin\frac{x}{2} dx = -2\sqrt{2}\int x d\cos\frac{x}{2} =$
$2\sqrt{2}\left(-x\cos\frac{x}{2} + \int \cos\frac{x}{2} dx\right) = 2\sqrt{2}\left(-x\cos\frac{x}{2} + 2\sin\frac{x}{2}\right) + C. \square$

(d) $\int \tan^n x dx = \int \tan^{n-2}(\sec^2 x - 1) dx = \int \tan^{n-2} d\tan x - \int \tan^{n-2} x dx =$
$\frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx. \square$

QUESTION 4. Evaluate the integrals

(a) $\int \frac{x^2 dx}{\sqrt{1+x^2}}$;

(b) $\int \frac{dx}{x\sqrt{x^2-1}}$;

(c) $\int_0^{\pi/6} \sqrt{1+\sin x} dx$

SOLUTIONS.

(a) Split the integral $I = \int \frac{x^2 dx}{\sqrt{1+x^2}} = \int \sqrt{1+x^2} dx - \int \frac{dx}{\sqrt{1+x^2}} = I_1 - I_2$ and calculate I_1 and I_2 separately by the trigonometric substitution $x = \tan t$:

$$\begin{aligned} I_1 &= \int \sqrt{1+x^2} dx = \int \sec^3 t dt = \int \sec t d\tan t = \sec t \tan t - \int \tan t d\sec t \\ &= \sec t \tan t - \int \tan^2 t \sec t dt = \sec t \tan t - \int (\sec^2 t - 1) \sec t dt \\ &= \sec t \tan t + \ln|\sec t + \tan t| - I_1 \Longrightarrow \\ I_1 &= \frac{1}{2} \sec t \tan t + \frac{1}{2} \ln|\sec t + \tan t| + C = \frac{1}{2} x\sqrt{1+x^2} + \frac{1}{2} \ln|x + \sqrt{1+x^2}| + C. \\ I_2 &= \int \frac{dx}{\sqrt{1+x^2}} = \int \sec t dt = \ln|\sec t + \tan t| + C = \ln|x + \sqrt{1+x^2}| + C \Longrightarrow \\ I &= I_1 - I_2 = \frac{1}{2} x\sqrt{1+x^2} - \frac{1}{2} \ln|x + \sqrt{1+x^2}| + C. \square \end{aligned}$$

(b) The standard trigonometric substitution $x = \sec t$ gives

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\tan t \sec t}{\sec t \tan t} dt = \int dt = t + C = \sec^{-1} x + C = \arccos \frac{1}{x} + C. \square$$

(c) Put $t = \frac{\pi}{2} - x$, so that $\sqrt{1+\sin x} = \sqrt{1+\cos t} = \sqrt{2} \cos \frac{t}{2}$, and

$$\int_0^{\pi/6} \sqrt{1+\sin x} dx = -\sqrt{2} \int_{\pi/2}^{\pi/3} \cos \frac{t}{2} dt = -2\sqrt{2} \sin \frac{t}{2} \Big|_{\pi/2}^{\pi/3} = 2 - \sqrt{2}. \square$$

QUESTION 5. Evaluate the integrals

(a) $\int \frac{dx}{x^3 + x^2 - 2x}$;

(b) $\int \frac{x^4 + 1}{x(x^2+1)^2} dx$.

and investigate convergence of the improper integrals

(c) $\int_0^\infty \sin(x^2) dx$;

Hint: Change the variable and integrate by parts.

(d) $\int_1^\infty \frac{x^3 dx}{x^4 + 2x^2 + 6}$.

SOLUTIONS.

(a)-(b) Just make use of the partial fraction decompositions

$$\begin{aligned} \int \frac{dx}{x^3 + x^2 - 2x} &= \int \left[-\frac{1}{2x} + \frac{1}{3(x-1)} + \frac{1}{6(x+2)} \right] dx = -\frac{1}{2} \ln x + \frac{1}{3} \ln(x-1) + \frac{1}{6} \ln(x+2) + C; \\ \int \frac{x^4 + 1}{x(x^2+1)^2} dx &= \int \left[\frac{1}{x} - \frac{2x}{(1+x^2)^2} \right] dx = \int \frac{dx}{x} - \int \frac{d(1+x^2)}{(1+x^2)^2} = \ln x + \frac{1}{1+x^2} + C. \square \end{aligned}$$

(c) Put $u = x^2, dx = \dfrac{du}{2\sqrt{u}}$ and transform the integral by parts

$$\begin{aligned}\int_0^\infty \sin(x^2)dx &= \frac{1}{2}\int_0^\infty \frac{\sin u}{\sqrt{u}}du = \frac{1}{2}\int_0^\infty \frac{d(1-\cos u)}{\sqrt{u}} = \int_0^\infty \frac{d(\sin^2\frac{u}{2})}{\sqrt{u}} \\ &= \left.\frac{\sin^2\frac{u}{2}}{\sqrt{u}}\right|_0^\infty + \frac{1}{2}\int_0^\infty \frac{\sin^2\frac{u}{2}}{u\sqrt{u}}du.\end{aligned}$$

The substitution vanishes at infinity and the integral converges by comparison with convergent integral $\int_1^\infty \frac{du}{u\sqrt{u}}$. No singularity at zero.

The actual value of this Fresnel integral is $\dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$. $\square$

(d) The integral diverges at infinity by the *limit comparison test* with the divergent integral $\displaystyle\int_1^\infty \frac{dx}{x}$:

$$\lim_{x\to\infty}\left(\frac{x^3}{x^4+2x^2+6} : \frac{1}{x}\right) = 1. \square$$