MATH101: HOMEWORK IV: Spring 2011
For all Sections
(Due April 11th week: first hour of the last lecture day)

QUESTIONS:
1. Evaluate the following indefinite integrals:
(1.a)

$$\int \frac{1}{x^3}\sqrt{\frac{1-x^2}{x^2}}dx$$

(1.b)

$$\int \frac{dy}{\sqrt{e^{2y}-1}}$$

SOLUTION (1.a) Let $u = \frac{1-x^2}{x^2}$. Then we get $du = -\frac{2}{x^3}\,dx$ which leads to

$$\int \frac{1}{x^3}\sqrt{\frac{1-x^2}{x^2}}dx = -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{3}\,(\frac{1-x^2}{x^2})^{3/2} + C$$

(1.b) We write the integral as

$$\int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{e^{-y}\,dy}{\sqrt{1-e^{-2y}}}$$

Now let $u = e^{-y}$ we get

$$\int \frac{e^{-y}\,dy}{\sqrt{1-e^{-2y}}} = -\int \frac{du}{\sqrt{1-u^2}} = -\sin^{-1}(e^{-y}) + C$$

(2). Evaluate the following definite integrals:
(2.a)

$$\int_0^{\pi/4} \pi^{\tan x}\,\sec^2 x\,dx$$

(2.b)

$$\int_{\pi/4}^{3\pi/4} \csc^2 x dx$$

SOLUTION: **(2.a)** Let $u = \pi^{\tan x}$. Then $du = \ln\pi\,\pi^{\tan x}\,\sec^2 x\,dx$. Hence we get

$$\int_0^{\pi/4} \pi^{\tan x}\,\sec^2 x\,dx = \frac{1}{\ln\pi}\int du = \frac{1}{\ln\pi}\,\pi^{\tan x}|_0^{\pi/4} = \frac{1}{\ln\pi}[\pi - 1]$$

(2.b)

$$\int_{\pi/4}^{3\pi/4} \csc^2 x dx = -cotanx|_{\pi/4}^{3\pi/4} = 2$$

3. **(3.a)** Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right by the line $x = 3 - y$.

(3.b) Graph the function $f(x) = \sqrt{1-x}$ for $0 \le x < 1$ and $f(x) = (7x-6)^{-1/3}$ for $1 \le x \le 2$ and integrate it over its domain.
SOLUTION: **(3.a)** Let R be the region under consideration, then

$$A(R) = \int_0^1 2\sqrt{y}\, dy + \int_1^2 [3 - y - (y-1)^2] dy = \frac{4}{3} + \frac{7}{6} = \frac{5}{2}$$

(3.b)

$$\int_0^2 f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^2 f(x) dx = \int_0^1 \sqrt{1-x} dx + \int_1^2 (7x-6)^{-1/3} dx$$

$$= -\frac{2}{3}(1-x)^{3/2}|_0^1 + \frac{3}{14}(7x-6)^{2/3} = \frac{2}{3} + \frac{3}{14}(8^{2/3} - 1) = \frac{55}{42}$$

4. **(4.a)** Find the limit $\lim_{n\to\infty}\left(\frac{n}{n+1} + \frac{n}{n+2^2} + \cdots + \cdots + \frac{n}{n^2+k^2} + \cdots + \frac{n^2}{2n^2}\right)$.
(4.b) Find the volume of the following solid: The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by the planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.
SOLUTION: **(4.a)**

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+k^2} + \cdots + \frac{n}{2n^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+1/n^2} + \frac{1}{1+2^2/n^2} + \cdots + \frac{1}{1+k^2/n^2} + \cdots + \frac{1}{1+n^2/n^2}\right)$$

$$= \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x|_0^1 = \frac{\pi}{4}$$

(4.b)

$$V = \int_{-1}^1 A(y)\, dy = \int_{-1}^1 2x^2 dy = 2\int_{-1}^1 (1-y^2) dy = \frac{8}{3}$$

5. **(5.a)** Find the are of the curves $y^2 - x - 4y = 0$ and $y^2 + x - 2y = 0$. **(5.b)** Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.
SOLUTION: **(5.a)** Let R be the region between the curves,then

$$A(R) = \int_0^3 [2y - y^2 - (y^2 - 4y)] dy = 9$$

(5.b) Using the cylindrical shell method we get

$$V = 2\pi \int_0^1 (2-x)(x-x^2) dx = \frac{\pi}{2}$$