MATH 101: HOMEWORK 3: Spring 2011
For all Sections
(Due on the week of March 28: first hour of the last lecture day)

1a. Let $f(x) = x^{2/3}(x^2 - 4)$. Find the open intervals on which f is increasing and decreasing and identify the extrema of f and the points where they occur.

SOLUTION: One has $f'(x) = \frac{8}{3}x^{-1/3}(x^2 - 1)$, and the critical points are $x = \pm 1$ (with $f' = 0$) and $x = 0$ (with $f' = \infty$). The derivative is positive at $x \to +\infty$ and it changes sign at each critical point (as all exponents are odd). Thus, f is increasing on $(-1, 0)$ and $(1, +\infty)$ and decreasing on $(-\infty, -1)$ and $(0, 1)$; the local maximum is $f(0) = 0$, and the local minima are $f(-1) = f(1) = -3$.

1b. Let $f(x) = x^{2/3}(x - 5)$. Find the domain, possible symmetries, intervals of increasing and decreasing, critical points, extrema, intervals of concavity, points of inflection, and asymptotes. Sketch the graph.

SOLUTION: The domain is all real numbers. There are no symmetries (the function is neither even nor odd). The derivatives are:

$$f' = \frac{5}{3}x^{-1/3}(x - 2), \qquad f'' = \frac{10}{9}x^{-4/3}(x + 1).$$

From the first derivative: the critical points are $x = 2$ (with $y' = 0$) and $x = 0$ (with $y' = \infty$); the derivative is positive at $x \to +\infty$ and changes sign at each critical point. Hence, the function is increasing on $(-\infty, 0)$ and $(2, +\infty)$ and decreasing on $(0, 2)$; it has one local minimum $f(2) = -3\sqrt[3]{4} \approx -4.8$ and one local maximum $f(0) = 0$.

From the second derivative: one has $y'' = 0$ at $x = -1$ and y'' does not exist at $x = 0$. The second derivative is positive at $x \to +\infty$, it changes sign at $x = -1$, but it does **not** change sign at $x = 0$. Hence, the graph is concave down in $(-\infty, -1)$ and concave up in each of the intervals $(-1, 0)$ and $(0, +\infty)$. The only point of inflection is $x = -1$, $y = f(-1) = -6$.

The graph has no asymptotes: no vertical asymptotes since the function is continuous everywhere, and no oblique asymptotes since at $x \to \pm\infty$ the function grows like $x^{5/3}$, which is not linear.

The graph is shown in the figure. (Note that there is an inflection point at $(-1, -6)$ which `Maple` does not show well.)

2a. Find the limit $\lim_{x \to +\infty} \left(\frac{x^2 + 1}{x + 2} \right)^{1/x}$.

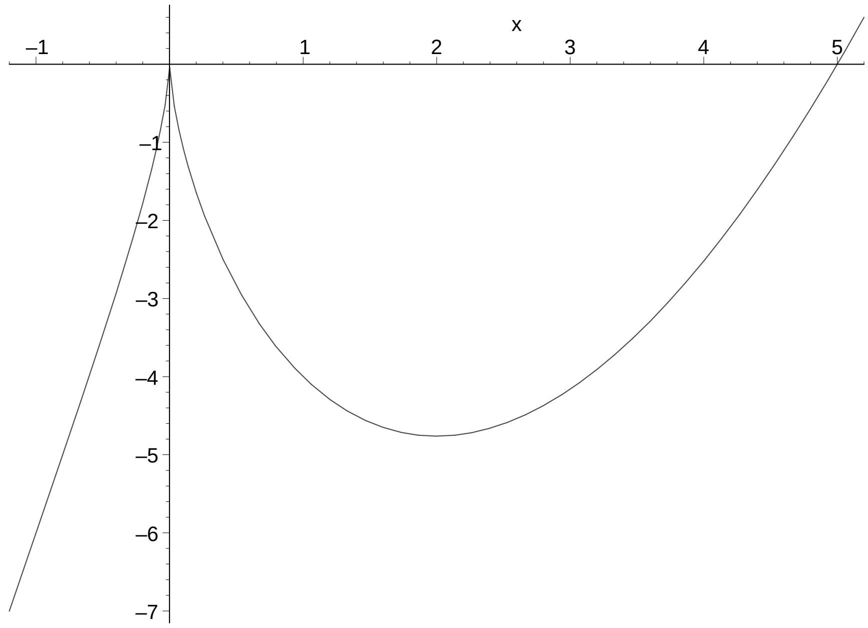
SOLUTION: Denote the limit in question by A. Then

$$\ln A = \lim_{x \to +\infty} \frac{\ln(x^2 + 1) - \ln(x + 2)}{x} \overset{*}{=} \lim_{x \to +\infty} \left(\frac{2x}{x^2 + 1} - \frac{1}{x + 2} \right) = 0$$

(where $*$ indicates L'Hôpital's rule) and $A = e^0 = 1$.

2b. Find the values of parameters a and b such that

$$\lim_{x \to 0} \left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x} \right) = 0.$$



Solution: First, note that $\lim_{x\to 0} \dfrac{\sin bx}{x} = b$ (with or without L'Hôpital's rule). For the other two terms, we have

$$\lim_{x\to 0}\left(\frac{\tan 2x}{x^3} + \frac{a}{x^2}\right) = \lim_{x\to 0}\frac{\tan 2x + ax}{x^3} \overset{*}{=} \lim_{x\to 0}\frac{2\sec^2 2x + a}{3x^2} = \lim_{x\to 0}\frac{2 + a\cos^2 2x}{3x^2\cos^2 2x}$$

Since $3x^2 \to 0$ and $\cos 2x \to 1$, this limit is infinity unless $2 + a\cos^2 2x \to 0$, which is the case if and only if $a = -2$. Assuming $a = -2$, we can apply L'Hôpital's rule two more times:

$$\lim_{x\to 0}\frac{2 - 2\cos^2 2x}{3x^2} \overset{*}{=} \lim_{x\to 0}\frac{8\cos 2x\sin 2x}{6x} = \lim_{x\to 0}\frac{4\sin 2x}{3x} \overset{*}{=} \lim_{x\to 0}\frac{8\cos 2x}{3} = \frac{8}{3}.$$

Hence, $a = -2$ and $b = -8/3$.

3. The stiffness S of a rectangular beam is proportional to its width times the cube of its height. Find the dimensions of the stiffest beam that can be cut from a cylindrical log of diameter d.

Solution: Let w and h be the width and the height of the beam, respectively. Then $w^2 + h^2 = d^2$ and, hence, $S(h) = h^3\sqrt{d^2 - h^2}$. Strictly speaking, this expression makes sense on the open interval $h \in (0, d)$ only, but we extend it to the closed segment $[0, d]$. Thus, we need to find the absolute maximum of the function $S(h) = h^3\sqrt{d^2 - h^2}$ on $[0, d]$.

Compute the derivative $S' = h^2(3d^2 - 4h^2)/\sqrt{d^2 - h^2}$ and find the critical points: $h = 0$, $h = d$, and $h = d\sqrt{3}/2$. (The other two points $h = -d$ and $h = -d\sqrt{3}/2$ are not in the interval.) The endpoints 0 and d are already among the critical points. Compute the values: $S(0) = S(d) = 0$ and $S(d\sqrt{3}/2) = 3d^4\sqrt{3}/16$, and compare; the maximum is attained at $h = d\sqrt{3}/2$ (and hence $w = d/2$).

4a. Use known formulas for areas to evaluate $\displaystyle\int_{-4}^{0} \sqrt{16 - x^2}\, dx$.

Solution: The integral in question represents the area of one quoter of the disk of radius $4 = \sqrt{16}$ about the origin (the quoter that is in the quadrant $x \le 0$, $y \ge 0$). Hence, it equals $\frac{1}{4}\pi(4)^2 = 4\pi$.

4b. Find the values $a < b$ that minimize the integral $I(a, b) := \displaystyle\int_a^b (x^4 - 2x^2)dx$.

Solution: The integrant $x^4 - 2x^2 = x^2(x^2 - 2)$ is negative for $-\sqrt{2} < x < \sqrt{2}$ and positive for $x < -\sqrt{2}$ or $x > \sqrt{2}$. Hence, it is obvious geometrically that $I(a, b)$ takes its minimal value, **which is negative**, at $(a, b) = (-\sqrt{2}, \sqrt{2})$. For a formal proof, observe that I is positive whenever $a < b \le -\sqrt{2}$ or $\sqrt{2} \le a < b$, and for the other pairs, we can estimate I using the additivity property. For example, if $a \le -\sqrt{2} \le b < \sqrt{2}$, one has

$$I(a, b) = I(a, -\sqrt{2}) + I(-\sqrt{2}, \sqrt{2}) - I(b, \sqrt{2}) > I(-\sqrt{2}, \sqrt{2})$$

since $I(a, -\sqrt{2}) \ge 0$ and $I(b, \sqrt{2}) > 0$ by the domination property.

5a. Compute $\displaystyle\int_{-1}^{1} (x^2 - 2x + 3)dx$.

Solution:

$$\int_{-1}^{1} (x^2 - 2x + 3)dx = \left(\frac{x^3}{3} - x^2 + 3x\right)\Bigg|_{-1}^{1} = \frac{20}{3}.$$

5b. Compute $\displaystyle\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)dx$.

Solution: We have $\cos x \ge 0$ for $x \in [0, \pi/2]$ and $\cos x \le 0$ for $x \in [\pi/2, \pi]$.

$$\begin{aligned}
&\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)dx \\
&\qquad = \int_0^{\pi/2} \frac{1}{2}(\cos x + \cos x)dx + \int_{\pi/2}^{\pi} \frac{1}{2}(\cos x - \cos x)dx \\
&\qquad\qquad = \int_0^{\pi/2} \cos x\, dx = -\sin x\Big|_0^{\pi/2} = 1.
\end{aligned}$$