Math 101, All sections, Homework #2 (Spring 2011)

Due to the week starting February 28, at the first hour of the last lecture day that week.

Question 1.

(a) Find an equation of the tangent to the curve $y = \sqrt{x}$ at the point $(x_0, y_0) = (1, 1)$.

(b) The height above the ground of a stone dropped by Galileo at $t = 0$ from the top of the Leaning Tower of Pisa varies with the time t measured in seconds by the equation $h = 56 - 4.9t^2$ meters. What is the speed of the stone (i) after two seconds; (ii) at the moment the stone hits the ground.

Solution. (a) Equation of the tangent line $(Y - y_0) = y'(x_0)(Y - x_0)$ with $y' = (\sqrt{x})' = \frac{1}{2\sqrt{x}} = \frac{1}{2}$ at $x = 1$ turns into $Y - 1 = \frac{1}{2}(X - 1)$. $\square$

(b) Speed is the absolute value of the velocity $\frac{dh}{dt} = -9.8t$. Hence (i) the speed is equal to $|-9.8 \cdot 2| = 19.6$ m/sec after 2 seconds and (ii) it is equal to $\left|-9.8 \cdot \sqrt{\frac{56}{4.9}}\right| \sim 33.13$ m/sec at the ground. The square root comes from the equation $h = 56 - 4.9 \cdot t^2 = 0$ that gives the time $t = \sqrt{\frac{56}{4.9}}$ the stone hits the ground. $\square$

Question 2. Apply the differentiation rules to find the derivatives of the following functions

(a) $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$;

(b) $r = \dfrac{e^t}{t}$;

(c) $u = \sin^{-1}(\ln x)$;

(d) v=ln$(\tan x)$.

Solution. These are mostly problems on the chain rule.

(a) Put $u = \sqrt{s}$ and apply the chain rule

$$f'(s) = \left(\frac{u-1}{u+1}\right)_u \cdot u' = \frac{2u'}{(u+1)^2} = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2},$$

where the index u indicates that the derivative should be taken with respect to variable u. $\square$

(b) Just apply the quotient rule $r' = \left(\frac{e^t}{t}\right)' = \frac{e^t t - e^t}{t^2} = e^t \frac{t-1}{t^2}$. $\square$

(c) The chain rule again: $u' = (\sin^{-1})'(\ln x) \cdot (\ln x)' = \frac{1}{\sqrt{1+\ln^2 x}} \cdot \frac{1}{x}$. $\square$

(d) More of the same: $[\ln(\tan x)]' = \frac{1}{\tan x} \cdot \frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x}$. $\square$

Question 3. Make use of the logarithmic derivative to differentiate the following functions:

(a) $y = t^{\sqrt{t}}$;

(b) $y = (\sin x)^x$.

Solution.

(a) Write $y = e^{\sqrt{t}\ln t}$ and apply the chain rule

$$y' = e^{\sqrt{t}\ln t}(\sqrt{t}\ln t)' = t^{\sqrt{t}}\left(\frac{1}{2\sqrt{t}}\ln t + \sqrt{t} \cdot \frac{1}{t}\right) = t^{\sqrt{t}} \cdot \frac{\ln t + 2}{2\sqrt{t}}. \quad \square$$

(b) Take logarithm $\ln y = x\ln(\sin x)$ and differentiate both sides

$$\frac{y'}{y} = \ln\sin x + x \cdot \frac{\cos x}{\sin x} \Longrightarrow y' = (\sin x)^x\left(\ln\sin x + x\cot x\right). \quad \square$$

Question 4. Find derivatives $\frac{dy}{dx}$ of the functions given by implicit equations

(a) $\ln xy = e^{x+y}$;

(b) $x^y = y^x$.

Solution.

(a) Rewrite the equation as $\ln x + \ln y = e^{x+y}$, differentiate both sides $\frac{1}{x} + \frac{y'}{y} = e^{x+y}(1 + y')$ and solve for the derivative y'

$$y' = \frac{e^{x+y} - \frac{1}{x}}{\frac{1}{y} - e^{x+y}} = \frac{xye^{x+y} - y}{x - xye^{x+y}}. \quad \square$$

(b) Take logarithm $y\ln x = x\ln y$, differentiate $y'\ln x + \frac{y}{x} = \ln y + \frac{x}{y}y'$, and solve for the derivative

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}. \quad \square$$

Question 5. Show that

(a) $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$;

(b) The function $y = \sin(\ln x)$ satisfies the equation $x^2y'' + xy' + y = 0$.

SOLUTION.

(a) Assume that $x \neq 0$ (otherwise both sides $= 1$), and put $t = \frac{n}{x}$. Then

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{tx} = \left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right]^x = e^x. \quad \square$$

(b) Indeed, by the chain rule $y' = [\sin(\ln x)]' = \frac{\cos(\ln x)}{x}$ which gives the second term of the equation $xy' = \cos(\ln x)$. To calculate the first term we need the second derivative

$$y'' = \left[\frac{\cos(\ln x)}{x}\right]' = \frac{-\sin(\ln x)\cdot\frac{1}{x}\cdot x - \cos(\ln x)}{x^2}$$

that yields $x^2y'' = -\sin(\ln x) - \cos(\ln x) = -y - xy'$ and $x^2y'' + xy' + y = 0$.