

# MATH101: HOMEWORK I: Spring 2011

## Solutions

(Due February 14th week: first hour of the last lecture day)

### QUESTIONS:

1. (a) prove that

$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

does not exist.

**Solution:**

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

,

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^{+-}} \frac{-x}{x} = \lim_{x \rightarrow 0^+} -1 = -1,$$

Hence

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} \neq \lim_{x \rightarrow 0^-} \frac{|x|}{x}$$

Then the limit

$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

does not exist.

(b). Show that

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin\left(\frac{\pi}{x}\right) = 0$$

**Solution:** Using the sandwich theorem we have

$$-\sqrt{x^3 + x^2} \leq \sqrt{x^3 + x^2} \sin\left(\frac{\pi}{x}\right) \leq \sqrt{x^3 + x^2}$$

For all  $x$ . Since  $\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} = 0$  then

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin\left(\frac{\pi}{x}\right) = 0$$

2. (a). If  $1 \leq f(x) \leq x^2 + 2x + 2$  for all  $x$  find

$$\lim_{x \rightarrow -1} f(x)$$

**Solution:** Using the sandwich theorem we get simply

$$\lim_{x \rightarrow -1} f(x) = 1$$

(b). Show that

$$\lim_{x \rightarrow 0} \sqrt{x} e^{\cos \frac{\pi^2}{x}} = 0$$

**Solution:** Since  $-1 \leq \cos \frac{\pi^2}{x} \leq 1$  then

$$e^{-1} \leq e^{\cos \frac{\pi^2}{x}} \leq e$$

Then using the sandwich theorem we have

$$e^{-1} \sqrt{x} \leq \sqrt{x} e^{\cos \frac{\pi^2}{x}} \leq \sqrt{x} e$$

Since  $\lim_{x \rightarrow 0} \sqrt{x} = 0$  then

$$\lim_{x \rightarrow 0} \sqrt{x} e^{\cos \frac{\pi^2}{x}} = 0$$

3. (a). Find

$$\lim_{x \rightarrow 1} \arcsin \frac{1 - \sqrt{x}}{1 - x}$$

**Solution:**

$$\begin{aligned} \lim_{x \rightarrow 1} \arcsin \frac{1 - \sqrt{x}}{1 - x} &= \arcsin \lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})}{1 - x} \\ &= \arcsin \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \arcsin \frac{1}{2} = \frac{\pi}{6} \end{aligned}$$

(b). Let

$$f(x) = \begin{cases} kx + 1, & x \leq 3 \\ kx^2 + k, & x > 3 \end{cases}$$

be a continuous function for all  $x$  values, where  $k$  is a real constant. Find  $k$ .

**Solution:**

$$\lim_{x \rightarrow 3^-} f(x) = 3k + 1$$

$$\lim_{x \rightarrow 3^+} f(x) = 10k$$

Hence  $3k + 1 = 10k$ , then  $k = 1/7$ .

4. Find the following limits

(a)

$$\lim_{x \rightarrow -\infty} \tan^{-1}(x^4 - 3x^3)$$

**Solution:**

$$\lim_{x \rightarrow -\infty} \tan^{-1}(x^4 - 3x^3) = \tan^{-1} \infty = \frac{\pi}{2}$$

(b)

$$\lim_{x \rightarrow \pm\infty} \ln(\sqrt{2x^2 + 4x} - \sqrt{2x^2 + 2x})$$

**Solution:**

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} \ln(\sqrt{2x^2 + 4x} - \sqrt{2x^2 + 2x}) &= \lim_{x \rightarrow \pm\infty} \ln\left[(\sqrt{2x^2 + 4x} - \sqrt{2x^2 + 2x}) \frac{(\sqrt{2x^2 + 4x} + \sqrt{2x^2 + 2x})}{(\sqrt{2x^2 + 4x} + \sqrt{2x^2 + 2x})}\right] \\ &= \lim_{x \rightarrow \pm\infty} \ln\left[\frac{2x}{(\sqrt{2x^2 + 4x} + \sqrt{2x^2 + 2x})}\right] = \ln\left[\pm \frac{1}{\sqrt{2}}\right] \end{aligned}$$

Hence limit  $x \rightarrow -\infty$  does not exist and

$$\lim_{x \rightarrow +\infty} \ln(\sqrt{2x^2 + 4x} - \sqrt{2x^2 + 2x}) = -\frac{1}{2} \ln 2$$

5. Suppose that a function  $f$  is continuous on the closed interval  $[0, 1]$  and that  $0 \leq f(x) \leq 1$  for every  $x$  in  $[0, 1]$ . Show that there exists a number  $c$  in  $[0, 1]$  such that  $f(c) = c$  ( $c$  is called a fixed point of  $f$ ).

**Solution:** let  $g(x) = f(x) - x$ . Since  $g(0) = f(0) \geq 0$  and  $g(1) = f(1) - 1 \leq 0$  Hence  $g(1) \leq 0 \leq g(0)$ . Since  $g(x)$  is a continuous function over the closed interval  $[0, 1]$ . Then by the intermediate value theorem there exists a point  $c$  in  $[0, 1]$  such that  $g(c) = 0$ . Hence there exists a point  $c$  in  $[0, 1]$  such that  $f(c) = c$ .