MATH101: HOMEWORK I: Spring 2011
Solutions
(Due February 14th week: first hour of the last lecture day)

QUESTIONS:
1. (a) prove that

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exists.
Solution:

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

,

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^{+-}} \frac{-x}{x} = \lim_{x\to 0^+} -1 = -1,$$

Hence

$$\lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}$$

Then the limit

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exists.
(b). Show that

$$\lim_{x\to 0} \sqrt{x^3 + x^2}\, \sin(\frac{\pi}{x}) = 0$$

Solution:Using the sandwich theorem we have

$$-\sqrt{x^3 + x^2} \leq \sqrt{x^3 + x^2}\, \sin(\frac{\pi}{x}) \leq \sqrt{x^3 + x^2}$$

For all x. Since $\lim_{x\to 0} \sqrt{x^3 + x^2} = 0$ then

$$\lim_{x\to 0} \sqrt{x^3 + x^2}\, \sin(\frac{\pi}{x}) = 0$$

2. (a). If $1 \leq f(x) \leq x^2 + 2x + 2$ for all x find

$$\lim_{x\to -1} f(x)$$

Solution: Using the sandwich theorem we get simply

$$\lim_{x\to -1} f(x) = 1$$

(b). Show that

$$\lim_{x\to 0} \sqrt{x}\, e^{\cos\frac{\pi^2}{x}} = 0$$

Solution: Since $-1 \le \cos\frac{\pi^2}{x} \le 1$ then

$$e^{-1} \le e^{\cos\frac{\pi^2}{x}} \le e$$

Then using the sandwich theorem we have

$$e^{-1}\sqrt{x} \le \sqrt{x}\, e^{\cos\frac{\pi^2}{x}} \le \sqrt{x}\, e$$

Since $\lim_{x\to 0}\sqrt{x} = 0$ then

$$\lim_{x\to 0}\sqrt{x}\, e^{\cos\frac{\pi^2}{x}} = 0$$

3. (a). Find

$$\lim_{x\to 1} \arcsin\frac{1-\sqrt{x}}{1-x}$$

Solution:

$$\lim_{x\to 1} \arcsin\frac{1-\sqrt{x}}{1-x} = \arcsin\lim_{x\to 1}\frac{(1-\sqrt{x})}{1-x}$$

$$= \arcsin\lim_{x\to 1}\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \arcsin\frac{1}{2} = \frac{\pi}{6}$$

(b). Let

$$f(x) = \begin{cases} kx+1, & x \le 3 \\ kx^2+k, & x > 3 \end{cases}$$

be a continuous function for all x values, where k is a real constant. Find k.
Solution:

$$\lim_{x\to 3^-} f(x) = 3k+1$$

$$\lim_{x\to 3^+} f(x) = 10k$$

Hence $3k+1 = 10k$, then $k = 1/7$.
4. Find the following limits
(a)

$$\lim_{x\to -\infty} \tan^{-1}(x^4 - 3x^3)$$

Solution:

$$\lim_{x\to -\infty} \tan^{-1}(x^4 - 3x^3) = tan^{-1}\infty = \frac{\pi}{2}$$

(b)

$$\lim_{x\to \pm\infty} \ln(\sqrt{2x^2+4x} - \sqrt{2x^2+2x})$$

Solution:

$$\lim_{x\to \pm\infty} \ln(\sqrt{2x^2+4x} - \sqrt{2x^2+2x}) = \lim_{x\to \pm\infty} \ln[(\sqrt{2x^2+4x} - \sqrt{2x^2+2x})\frac{(\sqrt{2x^2+4x} + \sqrt{2x^2+2x})}{(\sqrt{2x^2+4x} + \sqrt{2x^2+2x})}]$$

$$= \lim_{x\to \pm\infty} \ln[\frac{2x}{(\sqrt{2x^2+4x} + \sqrt{2x^2+2x})}] = \ln[\pm\frac{1}{\sqrt{2}}]$$

Hence limit $x \to -\infty$ does not exist and

$$\lim_{x\to+\infty} \ln(\sqrt{2x^2+4x} - \sqrt{2x^2+2x}) = -\frac{1}{2}\ln 2$$

5. Suppose that a function f is continuous on the closed interval $[0,1]$ and that $0 \le f(x) \le 1$ for every x in $[0,1]$. Show that there exists a number c in $[0,1]$ such that $f(c) = c$ (c is called a fixed point of f).

Solution: let $g(x) = f(x) - x$. Since $g(0) = f(0) \ge 0$ and $g(1) = f(1) - 1 \le 0$ Hence $g(1) \le 0 \le g(0)$. Since $g(x)$ is a continuous function over the closed integral $[0,1]$. Then by the intermediate value theorem there exists a point c in $[0,1]$ such that $g(c) = 0$. Hence there exists a point c in $[0,1]$ such that $f(c) = c$.