MATH. 346
DIFFERENTIAL GEOMETRY II
ASSIGNED EXERCISES III

Isometry, Infinitesimal Isometry and Killing Vectors

(1.a). **First Fundamental Form:**. The quadratic form $I_p(w) =< w, w >_p$, where $w \in T_p(S)$ of a regular surface S is called the first fundamnetal form of the regular surface $S \subset \mathbb{R}^3$ at $p \in S$ (see page 92 of do Carmo). Let $X(u, v)$ be a parametrization of S in the neighborhood of its point p where $(u, v) \in U \subset \mathbb{R}^2$. Any tangent vector $w \in T_p(S)$ is the tangent vector to a parametrized curve $\alpha(t) = X(u(t), v(t)),\ t \in (-\epsilon, \epsilon)$, with $p = \alpha(0) = X(u_0, v_0)$ and $w = \alpha'(0)$, we obtain

$$\begin{aligned} I_p(w) &= I_p(\alpha'(0)) =< X_u u' + X_v v', X_u u' + X_v v' >_p, \\ &= E\,(u')^2 + 2F\,u'\,v' + G(v')^2 \end{aligned} \tag{1}$$

(1.b). **Covariant Derivative**: Let $\alpha : I = [a, b] \to S$ be curve in S with parameter $s \in I$, where $\alpha(0) = p \in S$ and $\alpha'(0) = v \in T_p(S)$. Let w be a vector field along the curve α in S, so that $w = w^i\, X_{,i}$. Show that **Covariant derivative** of the vector field in the direction of v is given by

$$\frac{D\,w}{ds} = (\nabla_k\, w^i)\,\frac{du^k}{ds}\,X_{,i}$$

where

$$\nabla_k\, w^i = \frac{\partial w^i}{\partial u^k} + \Gamma^i_{kl}\, w^l, \quad k, i = 1, 2$$

We sometimes call the component w^i as the contravariant vector and $\nabla_k w^i$ as its covariant derivative.

(1.c). **Geodesics**: A curve $\alpha : I = [a, b] \to S$ in S with parameter $s \in I$ is a geodesic in S if it satisfies

$$\frac{D\,\alpha'}{ds} = 0,$$

This is equivalent to

$$\frac{d^2u^i}{ds^2} + \Gamma^i_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 0$$

(1.d). **Isometry**: (do Carmo page 218) A diffeomorphism $\varphi : S \to \bar{S}$ is an isometry if for all $p \in S$ and all pairs of vectors $w_1, w_2 \in T_p(S)$ we have

$$< w_1, w_2 >_p = < d\varphi(w_1), d\varphi(w_2) >_{\varphi(p)} \tag{2}$$

Exercise 1. Show that the above definition is equivalent to the following statement: A diffeomorphism $\varphi : S \to \bar{S}$ is an isometry if for all $p \in S$ and for all $w \in T_p(S)$ The firs fundamental form is preserved, i.e.,

$$I_p(w) = I_{\varphi(p)}(d\varphi_p(w)) \tag{3}$$

(1.e). **Infinitesimal Isometry**: An infinitesimal Isometry is defined as follows: Let $\phi_p = p + \epsilon\xi$ where $\epsilon << 1$ is a infinitesimally small and $\xi \in T_p(S)$. Hence Let $\alpha(s)$ be a curve parametrized by its arc length and $\alpha(0) = p$ and $w = \alpha'(0)$. Then under the infinitesimal isometry

$$\varphi(\alpha(s) = \alpha(s) + \epsilon\,\xi \tag{4}$$

Here ξ can be considered the vector field along the curve α. Furthermore show that

$$d\varphi(\alpha'(s)) = \alpha'(s) + \epsilon\,\xi' \tag{5}$$

Exercise 2. Show that under the infinitesimal isometry $I_p(w) = I_{\varphi(p)}(d\varphi_p(w))$ the vector ξ must satisfy (letting $\epsilon^2 = 0$)

$$< \alpha'(s), \xi' >= 0$$

for all for all curves α in S. This equation is called the Killing equation in S and the vector field ξ in S is called the Killing vector field of S (if it exists). Killing vectors are called the generators of the infinitesimal isometry mapping of surfaces.

Exercise 3. Show that the Killing equation can be written in index notation as: Let $\xi = \xi^k\, X_{,k}$ and $\xi_k = g_{kl}\, \xi^l$ then

$$\nabla_i\, \xi_j + \nabla_j\, \xi_i = 0$$

Exercise 3. **Killing vectors of the plane**. Let P be the plane $z = 0$ in $\mathbb{R}^2$. It is parametrized as $X(x, y),\ (x, y) \in \mathbb{R}^2$ such that $X_{xx} = 0, X_{xy} = 0, X_{yy} = 0$ because the christoffel symbol and the coefficient of second fundamental form vanish. Then any curve $\alpha : I \to S$ has the tangent vector $\alpha'(t) = X_{,x}\, x' + X_{,y}\, y'$ at any point $\alpha(t)$ of the curve. The killing vector ξ on S can be given as $\xi = X_{,x}\, \xi_1 + X_{,y}\, \xi_2$ where ξ_1 and ξ_2 are the components of the vector field with respect to the basis $\{X_{,x}, X_{,y}\}$ at an arbitrary point os S. Then

$$\xi' = \frac{d\xi_1}{dt}\, X_{,x} + \frac{d\xi_2}{dt}\, X_{,y}$$

Hence the Killing equation becomes

$$\begin{aligned} < \alpha'(s), \xi' > &= < X_{,x}\, x' + X_{,y}\, y', \frac{d\xi_1}{dt}\, X_{,x} + \frac{d\xi_2}{dt}\, X_{,y} >, \\ &= x'\, [E\, \xi_1' + F\, \xi_2'] + y'\, [F\, \xi_1' + G\, \xi_2'], \\ &= x'\, \xi_1' + y'\, \xi_2', \end{aligned} \tag{6}$$

where $E = G = 1, F = 0$ for the plane. We can write (by chain rule)

$$\frac{d\xi_i}{dy} = \xi_{i,x}\, x' + \xi_{1i,y}\, y', \quad i = 1, 2$$

Hence the Killing equation reduces to

$$\xi_{1,x}\, (x')^2 + (\xi_{1,y} + \xi_{2,x})\, x'\, y' + \xi_{2,y}\, (y')^2 = 0$$

Since the curve α is arbitrary the Killing equation for the plane becomes

$$\xi_{1,x} = 0, \quad \xi_{1,y} + \xi_{2,x} = 0, \quad \xi_{2,y} = 0$$

We can solve the above equations exactly

$$\xi_1 = \lambda_0\, y + a, \quad \xi_2 = -\lambda_0\, x + b,$$

where λ_0, a and b are real constant. Hence the infinitesimal isometry transformations become

$$x' = x + \epsilon\xi_1 = x + \lambda_0\, y + a,$$
$$y' = y + \epsilon\xi_2 = y - \lambda_0\, x + ab,$$

exponentiating the above infinitesimal transformations we get the finite ones

$$x' = x\, \cos\theta + y\, \sin\theta + a,$$
$$y' = y \cos\theta - x\, \sin\theta + b,$$

where $\theta = \epsilon\, \lambda_0$. This is the combination of rotation and translation in plane.

Exercise 4. Find the Killing vectors of cylinder and the sphere.

1.f. Commutator: Algebra of Killing Vectors. In a surface $S \in \mathbb{R}^3$ there can be at most 3 Killing vectors. It is the plane which has that number of Killing vectors. Above we have found the most general Killing vector of the plane containing three arbitrary parameters. Coefficients of each parameter are linearly independent Killing vectors of the plane. They are given as follows

$$\xi^1 = X_{,x}, \quad \xi^2 = X_y, \quad \xi^3 = y\, X_{,x} - x\, X_{,y} \tag{7}$$

These vectors form a bases of the infinitesimal isometry mappings. They are closed under a bracket $[\xi, \mu]$ called the commutator of the Killing vectors ξ and μ which is defined as follows (repeated indices are summed up from 1 to 2)

$$[\xi, \mu] \equiv [\alpha_i\, \partial_i\, \beta_k - \beta_i\, \partial_i\, \alpha_k]\, X_{,k} \tag{8}$$

where $\xi = \alpha_1\, X_{,x} + \alpha_2\, X_{,y}$ and $\mu = \beta_1\, X_{,x} + \beta_2\, X_{,y}$. Here $\partial_i = \frac{\partial}{\partial x^i}$ with $x^i = (x, y)$. Under this bracket Killing vectors form a closed algebra, .i.e.,

$$[\xi^1, \xi^2] = 0, \quad [\xi^1, \xi^3] = -\xi^2, \quad [\xi^2, \xi^3] = \xi^1 \tag{9}$$