MATH. 346
DIFFERENTIAL GEOMETRY II
ASSIGNED EXERCISES II

1. Gauss-Weingarten-Codazzi Equations

First Fundamental Form:. Let S be a regular surface and $X(u,v)$ be a parametrization of in the neighborhood of its point p where $(u,v) \in U \subset \mathbb{R}^2$. The normal vector N at P is orthogonal to X_u and X_v. Hence the Tangent space $T_p(S)$ at p is spanned by X_u and X_v. Then the components of the first fundamental form are given by

$$\begin{aligned} < X_u, X_u > &= E, \quad < X_u, N >= 0, \\ < X_u, X_v > &= F, \quad < X_v, N >= 0, \\ < X_v, X_v > &= G, \quad < N, N >= 1, \end{aligned}$$

The Gauss Equations:

$$\begin{aligned} X_{,uu} &= \Gamma^1_{11} X_{,u} + \Gamma^2_{11} X_{,v} + eN, \\ X_{,uv} &= \Gamma^1_{12} X_{,u} + \Gamma^2_{12} X_{,v} + fN, \\ X_{vv} &= \Gamma^1_{22} X_{,u} + \Gamma^2_{22} X_{,v} + gN, \end{aligned}$$

where

$$e =< X_{uu}, N >,\ f =< X_{uv}, N >,\ g =< X_{vv}, N >$$

are the coefficients of the **second fundamental form**, and

$$\begin{aligned} \Gamma^1_{11} &= \frac{G\,E_u - 2F\,F_u + F\,E_v}{2(EG - F^2)},\ \Gamma^2_{11} = \frac{2E\,F_u - E\,E_v - F\,E_u}{2(EG - F^2)}, \\ \Gamma^1_{12} &= \frac{E\,G_u - F\,E_v}{2(EG - F^2)}, \qquad\qquad \Gamma^2_{12} = \frac{E\,G_u - F\,E_v}{2(EG - F^2)}, \\ \Gamma^1_{22} &= \frac{2G\,F_v - G\,G_u - F\,G_v}{2(EG - F^2)},\ \Gamma^2_{22} = \frac{E\,G_v - 2F\,F_v + F\,G_u}{2(EG - F^2)}, \end{aligned}$$

are the **christoffel** symbols where $\Gamma^1_{12} = \Gamma^1_{21}$ and $\Gamma^2_{12} = \Gamma^2_{21}$

Weingarten equations:

$$N_{,u} = \frac{fF - eG}{EG - F^2}\, X_{,u} + \frac{eF - fE}{EG - F^2}\, X_{,v},$$
$$N_{,v} = \frac{gF - fG}{EG - F^2}\, X_{,u} + \frac{fF - gE}{EG - F^2}\, X_{,v},$$

(1.a). Show that **Gauss' equations** can be written as

$$X_{,ij} = \Gamma^k_{ij}\, X_{,k} + h_{ij}\, N, \quad i, j = 1, 2$$

where $X_1 = X_u$, $X_2 = X_u$, $X_{11} = X_{uu}$, $X_{12} = X_{uv}$, $X_{22} = X_{vv}$, $h_{11} = e$, $h_{12} = h_{21} = f$, $h_{22} = g$. Here repeated indices are summed from 1 to 2. The christoffel symbols are given by

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{im}\, (g_{mj,k} + g_{mk,j} - g_{jk,m}) \tag{1}$$

where $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$ are the components of the first fundamental form or *components of the metric tensor* and h_{ij} are the components of the second fundamental form, g^{ij} are the components of the inverse of matrix g_{ik}, i.e., $g^{im}\, g_{mk} = \delta^i_k$.

(1.b). Show that **Weingarten's equations** can be written as

$$N_{,i} = -g^{kl}\, h_{il}\, X_k, \quad i = 1, 2. \tag{2}$$

Here the sum is both on l and on k.

(1.c). Show that **Codazzi's equations** equations can be written as

$$\frac{\partial}{\partial u_l}\, h_{ij} - \Gamma^k_{il}\, h_{kj} = \frac{\partial}{\partial u_j}\, h_{il} - \Gamma^k_{ij}\, h_{kl}, \quad i, j, l = 1, 2 \tag{3}$$

or

$$\nabla_l\, h_{ij} = \nabla_j\, h_{il}, \quad i, j, l = 1, 2 \tag{4}$$

where ∇ is covariant derivative. Here $u^i = u, u^2 = v$.

(1.d). **Covariant Derivative**: Let $\alpha : I = [a, b] \to S$ be curve in S with parameter $s \in I$, where $\alpha(0) = p \in S$ and $\alpha'(0) = v \in T_p(S)$. Let w be a vector field along the curve α in S, so that $w = w^i X_{,i}$. Show that **Covariant derivative** of the vector field is given by

$$\frac{D\, w}{ds} = (\nabla_k\, w^i)\, \frac{du^k}{ds}\, X_{,i}$$

where

$$\nabla_k\, w^i = \frac{\partial w^i}{\partial u^k} + \Gamma^i_{kl}\, w^l, \quad k, i = 1, 2$$

(1.e). **Geodesics**: A curve $\alpha : I = [a, b] \to S$ in S with parameter $s \in I$ is a geodesic in S if it satisfies

$$\frac{D\, \alpha'}{ds} = 0,$$

Show that this is equivalent to

$$\frac{d^2 u^i}{ds^2} + \Gamma^i_{jk}\, \frac{du^j}{ds}\, \frac{du^k}{ds} = 0$$

Some special coordinate curves $\alpha(u) = X(u, v_0)$, $\beta(v) = X(u_0, v)$:
1. $F = 0$ (Orthogonal curves).
2. $F = 0, f = 0$ (Lines of curvature).
3. $f = 0$ (Conjugate curves).
4. $E = G = 0$ (Isotropic curves).
5. $e = g = 0$ (Asymptotic curves).