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Some Special Integrable Surfaces

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Abstract

We consider surfaces arising from integrable partial differential equations and from their deformations. Symmetries of the equation, gauge transformation of the corresponding Lax pair and spectral parameter transformations are the deformations which lead infinitely many integrable surfaces. We also study the integrable Willmore surfaces.

Surfaces corresponding to integrable equations are called integrable and a connection formula, relating integrable equations to surfaces, was first established by Sym [1], [2]. Here in this work we shall give a brief introduction (following our previous work [3]) of the recent status of the subject and give also some new results.

Let $F : \mathcal{U} \to \mathbb{R}^3$ be an immersion of a domain $\mathcal{U} \in \mathbb{R}^2$ into $\mathbb{R}^3$. Let $(u, v) \in \mathcal{U}$. The surface $F(u, v)$ is uniquely defined to within rigid motions by the first and second fundamental forms. Let $N(u, v)$ be the normal vector field defined at each point of the surface $F(u, v)$. Then the triple $\{F_u, F_v, N\}$ define a basis of $T_p(S)$, where S is the surface parameterized by $F(u, v)$ and p is a point in S, $p \in S$. The motion of the basis on S is characterized by the Gauss-Weingarten (GW) equations. The compatibility of these equations are the well-known Gauss-Mainardi-Codazzi (GMC) equations. The GMC equations are coupled nonlinear partial differential equations for the coefficients $g_{ij}(u, v)$ and $d_{ij}(u, v)$ of the first and second fundamental forms respectively. For certain particular surfaces these equations reduce to a single or to a system of integrable equations. The correspondence between the GMC equations and the integrable equations has been studied extensively, see for example [3].

Recently a more systematic approach to surfaces, GMC equations and integrable equations has been established by defining surfaces on Lie algebras and on their Lie Groups. In particular this approach provides an explicit relation between symmetries of integrable equations and surfaces in $\mathbb{R}^3$. The investigation of this relation between generalized symmetries and the associated surfaces in $\mathbb{R}^3$ is the main subject of this work. We have a new result indicating that the sphere, for a large class of differential equations, is the integrable surface corresponding to the some special gauge transformations (generalizing the Theorem 2.3 of Ref.[3]) and to some translational symmetries. We also give a connection with the integrable surfaces and Willmore surfaces in the last section.

Let us first give the connection between the integrable equations with the surface in $\mathbb{R}^3$

Theorem 1 (Fokas-Gelfand [4]) *Let $U(u, v; \lambda), V(u, v; \lambda), A(u, v; \lambda), B(u, v; \lambda) \in su(2)$ be differentiable functions of u, v and λ in some neighborhood of $\mathbb{R}^2 \times \mathbb{R}$. Assume that these functions satisfy*

$$U_v - V_u + [U, V] = 0,$$

and

$$A_v - B_u + [A,V] + [U,B] = 0$$

Define $\Phi(u,v;\lambda) \in SU(2)$ *and* $F(u,v;\lambda) \in su(2)$ *by the equations*

$$\Phi_u = U\,\Phi \quad , \quad \Phi_v = V\,\Phi,$$

and

$$F_u = \Phi^{-1}\,A\,\Phi \quad , \quad F_v = \Phi^{-1}\,B\,\Phi.$$

Then for each λ, $F(u,v;\lambda)$ *defines a 2-dimensional surface in* $\mathbb{R}^3$,

$$x_j = F_j(u,v;\lambda) \ , \ j = 1,2,3 \ , \quad F = i\sum_{k=1}^{3} F_k\,\sigma_k,$$

where σ_k *are the usual Pauli matrices. The first and second fundamental forms of* S *are*

$$\begin{aligned}&(ds_I)^2 = <A,A>\,du^2 + 2<A,B>\,dudv + <B,B>\,dv^2,\\ &(ds_{II})^2 = <A_u + [A,U], C>\,du^2 + 2<A_v + [A,V], C>\,dudv +\\ &<B_v + [B,V], C>\,dv^2,\end{aligned}$$

where $<A,B> = -\frac{1}{2}trace(AB)$, $|A| = \sqrt{<A,A>}$, *and* $C = \frac{[A,B]}{|[A,B]|}$. *A frame on this surface* S, *is*

$$\Phi^{-1}A\Phi \quad , \quad \Phi^{-1}B\Phi \quad , \quad \Phi^{-1}C\Phi.$$

The Gauss and mean curvatures of S *are given by* $K = det(G)$, $H = trace(G)$, *where* $G = g^{-1}\,b$.

Given U and V to find A and B from the equation $A_v - B_u + [A,V] + [U,B] = 0$ is in general a difficult task. However, there are some deformations which provide us A and B directly. These deformations are given as follows ,[5], [3], [1],[2],[9],[10].

1. Spectral parameter invariance of the equation. Historically this was the first deformation of integrable equations which gives a very nice connection with the integrable surfaces and it has first established by Sym [1], [2]. His connection formula is given by

$$A = \partial U \frac{\partial}{\lambda}, \quad B = \frac{\partial V}{\partial \lambda}, \quad F = \Phi^{-1}\,\frac{\partial \Phi}{\partial \lambda}.$$

2. Symmetries of integrable differential equations. Let δ denote an operation representing one of such symmetries. Then

$$A = \delta U, \quad B = \delta V, \quad F = \Phi^{-1}\,\delta\Phi.$$

δ may represent the classical Lie symmetries and (if integrable) the generalized symmetries of the nonlinear PDE.

3. Gauge symmetries of the Lax equation.

$$A = \frac{\partial M}{\partial u} + [M,U], \quad B = \frac{\partial M}{\partial v} + [M,V], \quad F = \Phi^{-1}\,M\,\Phi.$$

Here M is any traceless 2×2 matrix.

Any linear combination of these deformations give also new A, B and F. Hence we observe that there are infinitely many surfaces corresponding to deformations of an integrable differential equation. Among these surfaces we focus our attention to some special cases. For illustration we shall first give surfaces corresponding to some deformations of sine-Gordon equation.

Deformations of Sine-Gordon Surfaces

The sine-Gordon equation is given by

$$\frac{\partial^2 \theta}{\partial u \partial v} = \sin\theta, \tag{1}$$

where $\theta(u, v) \in \mathbb{R}$ and time is denoted by v. Define $U(u, v, \lambda)$, and $V(u, v, \lambda)$ by

$$U = \frac{i}{2}(-\theta_u \, \sigma_1 + \lambda \, \sigma_3) \ , \ V = \frac{i}{2\lambda}(\sin\theta\sigma_2 - \cos\theta\sigma_3). \tag{2}$$

Let φ be a symmetry of equation (1), i.e., let φ be a solution of

$$\partial^2 \varphi \frac{\partial}{u} \partial v = \varphi \cos\theta. \tag{3}$$

Then for each φ Theorem 2 (with $\alpha = 0, M = 0$) implies the surface constructed from

$$A = -\frac{i}{2} \frac{\partial\varphi}{\partial u} \sigma_1 \ , \ B = -\frac{i}{2\lambda}\varphi(\cos\theta \, \sigma_2 + \sin\theta \, \sigma_3). \tag{4}$$

Equation (1) is an integrable equation and hence it admits infinitely many symmetries usually referred as generalized symmetries. Indeed, there exists infinitely many explicit solutions of equation (3) in terms of θ and its derivatives.The first few are

$$\varphi := \theta_u \ , \ \theta_v \ , \ \theta_{uuu} + \frac{\theta_u^3}{2} \ , \ \theta_{vvv} + \frac{\theta_v^3}{2} \ , \ ... \tag{5}$$

We now give the surfaces corresponding to these generalized symmetries

Let S be the surface generated by a generalized symmetry of the sine-Gordon equation. That is, let S be the surface generated by U, V, A, B defined by equations (2)-(4). The first and second fundemental forms, the Gaussian and the mean curvatures of this surface are given by

$$ds_I^2 = \frac{1}{4}(\varphi_u^2 \, du^2 + \frac{1}{\lambda^2} \varphi^2 \, dv^2) \ , \ ds_{II}^2 = \frac{1}{2}(\lambda\varphi_u \sin\theta du^2 + \frac{1}{\lambda}\varphi\theta_v dv^2) \tag{6}$$

$$K=\frac{4\lambda^2\theta_v\sin\theta}{\varphi\varphi_u}\;,\;H=\frac{2\lambda(\varphi_u\theta_v+\varphi\sin\theta)}{\varphi\varphi_u} \tag{7}$$

Let S be the particular surface defined above lemma corresponding to $\varphi=\theta_v$. This surface is the sphere with

$$ds_I^2=\frac{1}{4}(\sin^2\theta\,du^2+\frac{\theta_v^2}{\lambda^2}\,dv^2)\;,\;ds_{II}^2=\frac{\lambda}{2}(\sin^2\theta\,du^2+\frac{\theta_v^2}{\lambda^2}\,dv^2) \tag{8}$$

$$K=4\lambda^2\quad,\quad H=4\lambda \tag{9}$$

We now consider different class of surfaces associated with solutions of the sine-Gordon equation. These are called the Weingarten surfaces. Surfaces where the Gauss and mean curvatures are related are called the Weingarten surfaces. Some deformations of the sine-Gordon equation lead to the linear Weingarten surfaces. Let S be the surface constructed from U and V defined by equations (2) and from $A=\mu\frac{\partial U}{\partial\lambda}+\frac{ip}{2}[\sigma_1,U]$, $B=\mu\frac{\partial V}{\partial\lambda}+\frac{ip}{2}[\sigma_1,V]$. This surface is a linear Weingarten surface and parallel to a space of negative constant curvature. The distance between these surfaces is $\frac{p}{4}$. The relation between the Gauss and mean curvatures are given by

$$(\mu^2+\lambda^2\,p^2)\,K-2\,p\,\lambda^2\,H+4\,\lambda^2=0. \tag{10}$$

Let K_0 and H_0 be the Gaussian and mean curvatures of a surface S_0 with constant curvature K_0 and let S be parallel to S_0 then (see [3])

$$K_0=\frac{K}{1-2\,a\,H+a^2\,K}\;,\;H_0=\frac{H-a\,K}{1-2\,a\,H+a^2\,K} \tag{11}$$

where a is a constant. Hence comparing the first equation above and (10) we find that $a=\frac{p}{4}$ and $K_0=-\frac{16\,\lambda^2}{3p^2+4\mu^2}$. Hence S is parallel to a surface S_0 with negative constant curvature. $\frac{p}{4}$ is the distance between the surfaces.

From the above example, deformations of integrable nonlinear partial differential equations lead to some special surfaces , like sphere, Weingarten surfaces. Recently [3] we studied several integrable partial differential equations like the modified Korteweg de Vries, Nonlinear Schrödinger, hyperbolic sine-Gordon. We have found higher degree Weingarten (quadratic and higher) surfaces and proved that the deformed surface to any constant gauge transformation is the sphere. It is possible to generalize this result. The following surfaces are spheres.

1. Any gauge transformation M with constant determinant, $detM$ = a positive constant. Since $F=\Phi^{-1}\,M\,\Phi$, then $detF=x_1^2+x_2^2+x_3^2=detM$.

2. Translational symmetries $\delta=\partial_u$ or $\delta=\partial_v$. In these cases the embedding function takes the form $F=\Phi^{-1}\,U\,\Phi$ or $\Phi^{-1}\,V\,\Phi$. Deformed surface is the sphere if $detU$ or $detV$ is a positive constant. Sine-Gordon is an example. For other examples see [3].

Willmore Surfaces

As a final example we shall consider the following surfaces which seem to have a nice connection with the Willmore surfaces.

Theorem 2. (Bobenko [6]) *Let U and V be given by*

$$U = \begin{pmatrix} \frac{1}{2}u_z & -Qe^{-u/2} \\ \frac{H}{2}e^{u/2} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -\frac{H}{2}e^{u/2} \\ \bar{Q}e^{-u/2} & \frac{u_{\bar{z}}}{2} \end{pmatrix},$$

satisfying the condition $U_{\bar{z}} - V_z + [U, V] = 0$ which is equivalent to

$$\begin{aligned} &u_{,z\bar{z}} + \tfrac{1}{2}H^2 e^u - 2Q\bar{Q}e^{-u} = 0, \\ &Q_{,\bar{z}} = \tfrac{1}{2}H_{,z}e^u, \quad \bar{Q}_{,z} = \tfrac{1}{2}H_{,\bar{z}}e^u. \end{aligned}$$

Then the associated surface is given by: The first and second fundamental forms are

$$\begin{aligned} &ds_I^2 = e^u\, dz d\bar{z}, \\ &ds_{II}^2 = Q dz^2 + H e^u dz d\bar{z} + \bar{Q} d\bar{z}^2. \end{aligned}$$

Gaussian ,K, and mean ,H, curvatures are respectively given by

$$\begin{aligned} &K = H^2 - 2Q\bar{Q}e^{-2u}, \\ &u_{,z\bar{z}} + \tfrac{1}{2}H^2 e^u - 2Q\bar{Q}e^{-u} = 0. \end{aligned}$$

The basis $\{F_{,z}, F_{,\bar{z}}, N\}$ at each point on the surface is given by

$$F_{,z} = -ie^{u/2}\Phi^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Phi, \quad F_{,\bar{z}} = -ie^{u/2}\Phi^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Phi,$$

$$N = -i\Phi^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi,$$

where Φ satisfies the linear equations $\Phi_{,z} = U\,\Phi$ and $\Phi_{,\bar{z}} = V\,\Phi$. The matrices A and B defined in Theorem 1 are given by

$$A = \begin{pmatrix} 0 & 0 \\ -ie^{u/2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -ie^{u/2} \\ 0 & 0 \end{pmatrix}.$$

Let us parameterize the matrix function Φ as

$$\Phi = \begin{pmatrix} \bar{\psi}_2 & \bar{\psi}_1 \\ -\psi_1 & \psi_2 \end{pmatrix}.$$

Then we find that

$$\begin{aligned} &det(\Phi) = \psi_1\,\bar{\psi}_1 + \psi_2\,\bar{\psi}_2 = e^{u/2}, \\ &\bar{\psi}_1\,\psi_{2,z} - \psi_2\,\bar{\psi}_{1,z} = -Q, \\ &\psi_{1,z} = p\,\psi_2, \\ &\psi_{2,\bar{z}} = -p\,\psi_1, \end{aligned}$$

where $p = \frac{1}{2} H e^{u/2}$. From the expressions for $F_{,z}$ and $F_{,\bar{z}}$ one can show that

$$x_1 - i x_2 = \int_C [(\bar{\psi}_2)^2 d\bar{z} - (\bar{\psi}_1)^2 dz],$$
$$x_3 = \int_C [\bar{\psi}_1 \psi_2 dz + \psi_1 \bar{\psi}_2 d\bar{z}].$$

This is the Weierstrass representation of a surface often used by Konopelchenko and his collaborators [7], [8]. Here C is a contour in the complex z-plane. Willmore surfaces arise from the variation of the following functional

$$W(S) = \int\int_S H^2 d\sigma = \int\int_{\mathcal{U}} \sqrt{detg}\, H^2 dz d\bar{z}.$$

Willmore surfaces extremize this functional and defined by the following Euler-Lagrange equations (called the Willmore equation) [11]

$$\nabla^2 H + 2H(H^2 - K) = 0,$$

where ∇^2 is the Laplace-Beltrami operator defined on the surface. This equation is highly nonlinear. In particular if one parameterizes S as the graph of a differentiable function f , then the Willmore equation becomes a fourth order nonlinear partial differential equation for f. Sphere and a special torus are exact solutions of the Willmore equation [11]. We observed that, except the sphere cases, none of the integrable surfaces studied in [3] are Willmore (their H and K do not satisfy the above Willmore equation). For the surfaces defined in Theorem 2 the Willmore equation reduces to

$$H_{,z\bar{z}} + 2Q\bar{Q} H e^{-u} = 0,$$

or

$$H_{,z\bar{z}} + H u_{,z\bar{z}} + \frac{1}{2} H^3 e^u = 0.$$

As a result any integrable Willmore surface in conformal gauge must satisfy the following equations

Gauss equation:

$$u_{,z\bar{z}} + \frac{1}{2} H^2 e^u - 2Q\bar{Q} e^{-u} = 0,$$

Codazzi equations:

$$Q_{,\bar{z}} = \frac{1}{2} H_{,z} e^u, \quad \bar{Q}_{,z} = \frac{1}{2} H_{,\bar{z}} e^u,$$

Willmore equations:

$$H_{,z\bar{z}} + H u_{,z\bar{z}} + \frac{1}{2} H^3 e^u = 0.$$

Exact solutions of the above equations are: (a) The minimal surfaces $H = 0$, $Q = 0$ and u is a harmonic function. (b) The sphere , $H = \lambda, K = \lambda^2$ where λ is a constant, $Q = 0$ and u satisfies the Liouville equation $u_{,z\bar{z}} + \frac{1}{2}\lambda^2 e^u = 0$ which can be solved exactly. (c) Developable Surfaces , $K = 0, Q = \bar{Q} = \frac{1}{2} H e^u$ and $H_{z\bar{z}} + \frac{1}{2} H^3 e^u = 0$. Here u is a constant on the surface. Similarity solutions of the cubic nonlinear equation for H can be solved exactly interms of the Jacobi elliptic functions. (d) Torus. In [11] Willmore mentions an exact special torus solution and mentions also his conjecture (the Willmore conjecture) that $W(S) \geq 2\pi^2$ for all tori. Explicit torus solution in the above conformal gauge and also other solutions will be communicated later.

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