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Hamiltonian equations in $\mathbb{R}^3$

Ahmet Ay, Metin Gürses, and Kostyantyn Zheltukhin[a)]
Department of Mathematics, Faculty of Sciences, Bilkent University, 06800 Ankara, Turkey

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The Hamiltonian formulation of $N=3$ systems is considered in general. The most general solution of the Jacobi equation in $\mathbb{R}^3$ is proposed. The form of the solution is shown to be valid also in the neighborhood of some irregular points. Compatible Poisson structures and corresponding bi-Hamiltonian systems are also discussed. Hamiltonian structures, the classification of irregular points and the corresponding reduced first order differential equations of several examples are given. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1619204]

I. INTRODUCTION

The Hamiltonian formulation of a system of dynamical equations is important not only in mathematics but also in physics and other branches of natural sciences. They in general describe conserved systems. Among all possible odd dimensional cases, the three dimensional dynamical systems have a unique position. The Jacobi equation in this case reduces to a single scalar equation for three components of the Poisson structure J. Due to this property $N=3$ dynamical systems attracted much research to derive new Hamiltonian systems.[6–12] More recently[1,2] a large class of solutions of the Jacobi equation in $\mathbb{R}^3$ was given. Poisson structures, in all dimensions, were also considered in Ref. 3. In this work, we consider a general solution of the Jacobi equation in $\mathbb{R}^3$. We find the compatible Poisson structures and give the corresponding bi-Hamiltonian systems. We give all explicit examples in a special section and Table I at the end.

Let us give necessary information about the Poisson structures in $\mathbb{R}^3$. A matrix $J=(J_{ij})$, $i,j=1,2,3$, defines a Poisson structure in $\mathbb{R}^3$ if it is skew-symmetric, $J_{ij}=-J_{ji}$, and its entries satisfy the Jacobi equation

$$J^{li}\partial_l J^{jk}+J^{lj}\partial_l J^{ki}+J^{lk}\partial_l J^{ij}=0, \tag{1}$$

where $i,j,k=1,2,3$. Here we use the summation convention, meaning that repeated indices are summed up. Let us introduce the following notations. For matrix J put $J_{12}=u$, $J_{31}=v$, $J_{23}=w$. Then the Jacobi equation (1) takes the form

$$u\partial_1 v-v\partial_1 u+w\partial_2 u-u\partial_2 w+v\partial_3 w-w\partial_3 v=0. \tag{2}$$

It can also be rewritten as

$$u^2\partial_1\frac{v}{u}+w^2\partial_2\frac{u}{w}+v^2\partial_3\frac{w}{v}=0. \tag{3}$$

[We assume that none of the functions u, v and w vanish. If any one of these functions vanishes then Eq. (2) becomes trivial for the remaining two variables; see Remark 1.]

We consider the general solution of the Jacobi equation (3) and show that it has the following form:

$$J^{ij}=\mu\epsilon^{ijk}\partial_k\Psi, \tag{4}$$

a)Electronic mail: zhelt@fen.bilkent.edu.tr

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where μ and Ψ are arbitrary differentiable functions of x_i, $i=1,2,3$ and ϵ^{ijk} is the Levi–Civita symbol. We also consider special solutions given by

$$u\partial_1 v - v\partial_1 u = 0,\ w\partial_2 u - u\partial_2 w = 0, \quad \text{which implies} \quad v\partial_3 w - w\partial_3 v = 0. \tag{5}$$

Such Poisson structures appear in many examples. We show that this special class of solutions belongs to the general form (4). We introduce these special solutions to study the irregular points of the Poisson structures. All the irregular points of the Poisson structure matrix J given in the examples,[1] we know so far, come from this special form. Hence they are also irregular points of the form (4) we give.

II. THE GENERAL SOLUTION

Assuming that $u \neq 0$, let $\rho = v/u$ and $\chi = w/u$; then Eq. (2) can be written as

$$\partial_1\rho - \partial_2\chi + \rho\partial_3\chi - \chi\partial_3\rho = 0. \tag{6}$$

This equation can be put in a more suitable form by writing it as

$$(\partial_1 - \chi\partial_3)\rho - (\partial_2 - \rho\partial_3)\chi = 0. \tag{7}$$

Introducing differential operators D_1 and D_2 defined by

$$D_1 = \partial_1 - \chi\partial_3, \quad D_2 = \partial_2 - \rho\partial_3, \tag{8}$$

one can write Eq. (7) as

$$D_1\rho - D_2\chi = 0. \tag{9}$$

Lemma 1: Let Eq. (9) be satisfied. Then there are new coordinates $\overline{x_1}, \overline{x_2}, \overline{x_3}$ such that

$$D_1 = \partial_{\overline{x_1}} \quad \text{and} \quad D_2 = \partial_{\overline{x_2}}. \tag{10}$$

Proof: If Eq. (9) is satisfied, it is easy to show that the operators D_1 and D_2 commute, i.e.,

$$D_1 \circ D_2 - D_2 \circ D_1 = 0.$$

Hence, by the Frobenius theorem (see Ref. 4, p. 40) there exist coordinates $\overline{x_1}, \overline{x_2}, \overline{x_3}$ such that the equalities (10) hold. □

The coordinates $\overline{x_1}, \overline{x_2}, \overline{x_3}$ are described by the following lemma.

Lemma 2: Let ζ be a common invariant function of D_1 and D_2, i.e.,

$$D_1\zeta = D_2\zeta = 0, \tag{11}$$

then the coordinates $\overline{x_1}, \overline{x_2}, \overline{x_3}$ of Lemma 1 are given by

$$\overline{x_1} = x_1, \quad \overline{x_2} = x_2, \quad \overline{x_3} = \zeta. \tag{12}$$

Moreover from (11) we get

$$\chi = \frac{\partial_1\zeta}{\partial_3\zeta}, \quad \rho = \frac{\partial_2\zeta}{\partial_3\zeta}. \tag{13}$$

Theorem 1: *All Poisson structures in $\mathbb{R}^3$, except at some irregular points, take the form (4), i.e., $J_{ij} = \mu\,\epsilon_{ijk}\,\partial_k\,\zeta$. Here μ and ζ are some differentiable functions in $\mathbb{R}^3$*

Proof: Using (13), the entries of matrix J, in the coordinates $\overline{x_1}, \overline{x_2}, \overline{x_3}$, can be written as

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$$u=\mu\partial_3\zeta,$$

$$v=\mu\partial_2\zeta, \tag{14}$$

$$w=\mu\partial_1\zeta.$$

Thus matrix J has the form (4) ($\Psi=\zeta$). □

Remark 1: So far we assumed that $u\neq 0$. If $u=0$ then the Jacobi equation becomes quite simpler, $v\partial_3 w-w\partial_3 v=0$, which has the simple solution $w=v\,\xi(x_1,x_2)$, where ξ is an arbitrary differentiable of x_1 and x_2. This class is also covered by the general solution (4) by letting Ψ independent of x_3.

A well known example of a dynamical system with a Poisson structure of the form (4) is the Euler equations.

Example 1: Consider the Euler equations (Ref. 4, pp. 397–398),

$$\dot{x}_1=\frac{I_2-I_3}{I_2I_3}x_2x_3,$$

$$\dot{x}_2=\frac{I_3-I_1}{I_3I_1}x_3x_1, \tag{15}$$

$$\dot{x}_3=\frac{I_1-I_2}{I_1I_2}x_1x_2,$$

where $I_1,I_2,I_3\in\mathbb{R}$ are some (nonvanishing) real constants. This system admits a Hamiltonian representation of the form (4). The matrix J can be defined in terms of function $\Psi=-\frac{1}{2}(x_1^2+x_2^2+x_3^2)$ and $\mu=1$, so

$$u=-x_3,$$

$$v=-x_2, \tag{16}$$

$$w=-x_1,$$

and $H=x_1^2/2I_1+x_2^2/2I_2+x_3^2/2I_3$.

Recently, a large set of solutions of the Jacobi equation (3) satisfying (5) was given in Ref. 1. For all such solutions the Darboux transformation and Casimir functionals were obtained; see Ref. 1.

Definition 1: For every domain $\Omega\in\mathbb{R}^3$ let $\mathbf{I}_a(\Omega)$ be the set of all solutions of (5) defined in Ω with $u(x)$, $v(x)$, and $w(x)$ being $C^1(\Omega)$.

Following Ref. 1 we have as follows.

Proposition 1: Let $\eta(x_1,x_2,x_3),\psi_i(x_i),\phi_i(x_i)$, $i=1,2,3$, be arbitrary differentiable functions defined in Ω. Then the functions

$$u(x)=\eta(x_1,x_2,x_3)\psi_1(x_1)\psi_2(x_2)\phi_3(x_3),$$

$$v(x)=\eta(x_1,x_2,x_3)\psi_1(x_1)\phi_2(x_2)\psi_3(x_3), \tag{17}$$

$$w(x)=\eta(x_1,x_2,x_3)\phi_1(x_1)\psi_2(x_2)\psi_3(x_3),$$

define a solution of Eq. (5) belonging to $\mathbf{I}_a(\Omega)$.

Definition 2: For every domain $\Omega\in\mathbb{R}^3$, let $\mathbf{I}_b(\Omega)$ be the set of all solutions of (5) defined in Ω where one of the functions $u(x)$, $v(x)$, and $w(x)$ is zero and the others are not identically zero in Ω.

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Following Ref. 1 we have Proposition 2.

Proposition 2: Let $\eta(x_1,x_2,x_3),\xi_i(x_i)$, $i=1,2,3$, *be arbitrary differentiable functions defined in* Ω. *Then the functions*

$$u(x)=0,\quad v(x)=\eta(x_1,x_2,x_3)\xi_2(x_2),\quad w(x)=\eta(x_1,x_2,x_3)\xi_1(x_1) \tag{18}$$

define a solution of Eq. (1) belonging to $\mathbf{I}_b(\Omega)$, $u=0$. *Similar solutions can be given in the case* $v=0$ *and the case* $w=0$.

Remark 2: All of the Poisson structures described in Ref. 1 have the form (4). For the Poisson structure J, given by (17), assume ψ_1, ψ_2, and ψ_3 to be nonvanishing and define $\mu=\eta(x_1,x_2,x_3)\psi_1(x_1)\psi_2(x_2)\psi_3(x_3)$ and

$$\Psi=\int^{x_1}\frac{\phi_1}{\psi_1}dx_1+\int^{x_2}\frac{\phi_2}{\psi_2}dx_2+\int^{x_3}\frac{\phi_3}{\psi_3}dx_3;$$

then J has form (4). For the Poisson structure J, given by (18), define $\mu=\eta(x_1,x_2,x_3)$ and $\Psi=\int^{x_1}\xi_1(x_1)+\int^{x_3}\xi_2(x_2)$; then J has form (4).

Let us give two examples of systems that admit a Hamiltonian representation described by the Proposition 1 and Proposition 2.

Example 2: Consider the Lotka–Voltera system,[8,9]

$$\begin{aligned}\dot{x}_1&=-abcx_1x_3-bc\mu_0x_1+cx_1x_2+c\nu x_1,\\ \dot{x}_2&=-a^2bcx_2x_3-abc\mu_0x_2+x_1x_2,\\ \dot{x}_3&=-abcx_2x_3-abc\nu_0x_3+bx_1x_3,\end{aligned} \tag{19}$$

where $a,b,c,\mu_0,\nu_0\in\mathbb{R}$ are constants.

The matrix J is given by

$$\begin{aligned}u&=cx_1x_2,\\ v&=-bcx_1x_3,\\ w&=-x_2x_3,\end{aligned} \tag{20}$$

and $H=abx_1+x_2-ax_3+\nu_0\ln x_2-\mu_0\ln x_3$.

Example 3: Consider the Lorenz system[8]

$$\begin{aligned}\dot{x}_1&=\frac{1}{2}x_2,\\ \dot{x}_2&=-x_1x_3,\\ \dot{x}_3&=x_1x_2.\end{aligned} \tag{21}$$

The matrix J is given by

$$\begin{aligned}u&=\tfrac{1}{4},\\ v&=0,\\ w&=-\tfrac{1}{2}x_1,\end{aligned} \tag{22}$$

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and $H=x_2^2+x_3^2$. Many other examples are given in Sec. III.

In the derivation of the general solution, Theorem 1, we assumed that one of the components of matrix J is different from zero. In addition our derivation is valid only in a neighborhood of a regular point of J (matrix $J\neq 0$ at this point). If $p\in\mathbb{R}^3$ is an irregular point where $u(p)=v(p)=w(p)=0$ it is not clear whether our solution is valid in a neighborhood of such a point. Here we shall show that the Poisson structures given by (4) preserve their form in the neighborhood of the following irregular points.

Lemma 3: The solution of the equation (1) defined in Proposition 1 and Proposition 2 and written in the form (4) preserve their form in the neighborhood of the irregular points, lines and planes in $\mathbb{R}^3$ *defined below*

(a) Irregular points. Let $p=(p_1,p_2,p_3)$ *be such that* $\phi_1(p_1)=\phi_2(p_2)=\phi_3(p_3)=0$ *and* $\psi_i(p_i)\neq 0$, $i=1,2,3$; *then p is an irregular point where the general form (4) is preserved.*

(b) Irregular lines or irregular planes. Let $p=(p_1,p_2,p_3)\in\mathbb{R}^3$ *be such that* $\eta(p_1,x_2,x_3)=0$ $[\eta(x_1,p_2,x_3)=0$ *or* $\eta(x_1,x_2,p_3)=0]$ *and* $\psi_i(p_i)\neq 0$, $i=1,2,3$; *then* $x_1=p_1$ $(x_2=p_2$ *or* $x_3=p_3)$ *is an irregular plane, where the general form (4) is preserved. Let* $x_1=p_1$, $x_2=p_2$ *be such that* $\eta(p_1,p_2,x_3)=0$ $[\eta(p_1,x_2,p_3)=0$ *or* $\eta(x_1,p_2,p_3)=0]$ *and* $\psi_i(p_i)\neq 0$, $i=1,2,3$ *then* $x_1=p_1$, $x_2=p_2$ $(x_1=p_1,x_3=p_3$ *or* $x_2=p_2,x_3=p_3)$ *is an irregular line, where the general form (4) is preserved.*

Proof: The solution given in Proposition 1 and Proposition 2 solves the following equations (without any division):

$$u\partial_1 v-v\partial_1 u=0,$$
$$-u\partial_2 w+w\partial_2 u=0, \tag{23}$$
$$v\partial_3 w-w\partial_3 v=0.$$

The general form (4), given in Remark 2, is also preserved at such points since we can define $\mu=\eta(x_1,x_2,x_3)\psi_1(x_1)\psi_2(x_2)\psi_3(x_3)$ and

$$\Psi=\int^{x_1}\frac{\phi_1}{\psi_1}dx_1+\int^{x_2}\frac{\phi_2}{\psi_2}dx_2+\int^{x_3}\frac{\phi_3}{\psi_3}dx_3,$$

or if one of the components of J is zero, assume $u=0$, we define $\mu=\eta(x_1,x_2,x_3)$ and $\Psi=\int^{x_1}\xi_1(x_1)+\int^{x_3}\xi_2(x_2)$. □

Example 4: For the Euler system considered in Example 1 the Poisson structure, given by (16), has irregular point $p=(0,0,0)$. The irregular point $p=(0,0,0)$ satisfies the conditions of Lemma 3, the functions $\Psi=-\frac{1}{2}(x_1^2+x_2^2+x_3^2)$, $\mu=1$ in terms of which the Poisson structure is given, are well defined in a neighborhood of $p=(0,0,0)$.

III. BI-HAMILTONIAN SYSTEM

In general the Darboux theorem states that (see Ref. 4), locally, all Poisson structures can be reduced to the standard one (a Poisson structure with constant entries). The above theorem, Theorem 1, resembles the Darboux theorem for $N=3$. All Poisson structures, at least locally, can be cast into the form (4). This result is important because the Darboux theorem is not suitable for obtaining multi-Hamiltonian systems in $\mathbb{R}^3$, but we will show that our theorem is effective for this purpose. Writing the Poisson structure in the form (4) allows us to construct bi-Hamiltonian representations of a given Hamiltonian system.

Definition 3: Two Hamiltonian matrices J and $\tilde{J}$ *are compatible, if the sum* $J+\tilde{J}$ *defines also a Poisson structure.*

The compatible Poisson structures can be used to construct bi-Hamiltonian equations.

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Definition 4: A Hamiltonian equation is said to be bi-Hamiltonian if it admits two Hamiltonian representations with compatible Poisson structures,

$$\frac{dx}{dt} = J\nabla H = \tilde{J}\nabla \tilde{H}, \tag{24}$$

where J and $\tilde{J}$ are compatible.

Lemma 4: Let Poisson structures J and $\tilde{J}$ have the form (4), so $J^{ij} = \mu \epsilon^{ijk}\partial_k \Psi$ and *$\tilde{J}^{ij} = \tilde{\mu}\epsilon^{ijk}\partial_k \tilde{\Psi}$. Then J and $\tilde{J}$ are compatible if and only if there exists a differentiable function $\Phi(\Psi, \tilde{\Psi})$ such that*

$$\tilde{\mu} = \mu \frac{\partial_{\tilde{\Psi}} \Phi}{\partial_{\Psi} \Phi}, \tag{25}$$

provided that $\partial_{\Psi} \Phi \equiv \partial\Phi/\partial\Psi \neq 0$ and $\partial_{\tilde{\Psi}} \Phi \equiv \partial\Phi/\partial\tilde{\Psi} \neq 0$.

This suggests that all Poisson structures in $\mathbb{R}^3$ have compatible pairs, because the condition (25) is not so restrictive on the Poisson matrices J and $\tilde{J}$. Such compatible Poisson structures can be used to construct bi-Hamiltonian systems.

Lemma 5: Let J be given by (4) and $H(x_1, x_2, x_3)$ is any differentiable function; then the Hamiltonian equation,

$$\frac{dx}{dt} = J\nabla H = -\mu\, \nabla\Psi \times \nabla H, \tag{26}$$

is bi-Hamiltonian with the second structure given by $\tilde{J}$ with entries

$$\begin{aligned} \tilde{u}(x) &= \tilde{\mu}\, \partial_3 g(\Psi(x_1 x_2 x_3), H(x_1, x_2, x_3)), \\ \tilde{v}(x) &= -\tilde{\mu}\, \partial_2 g(\Psi(x_1 x_2 x_3), H(x_1, x_2, x_3)), \\ \tilde{w}(x) &= \tilde{\mu}\, \partial_1 g(\Psi(x_1 x_2 x_3), H(x_1, x_2, x_3)), \end{aligned} \tag{27}$$

and $\tilde{H} = h(\Psi(x_1 x_2 x_3), H(x_1, x_2, x_3))$, $\tilde{\Psi} = g(\Psi(x_1, x_2, x_3), H(x_1, x_2, x_3))$, $\tilde{\mu} = \mu\,(\partial_{\tilde{\Psi}}\Phi/\partial_{\Psi}\Phi)$. Provided that there exist differentiable functions $\Phi(\Psi, \tilde{\Psi})$, $h(\Psi, H)$, and $g(\Psi, H)$ satisfying the following equation:

$$\frac{\partial g}{\partial \Psi}\frac{\partial h}{\partial H} - \frac{\partial g}{\partial H}\frac{\partial h}{\partial \Psi} = \frac{\Phi_1(\Psi, g)}{\Phi_2(\Psi, g)}, \tag{28}$$

where $\Phi_1 = \partial_{\Psi}\Phi|_{(\Psi, g)}$, $\Phi_2 = \partial_{\tilde{\Psi}}\Phi|_{(\Psi, g)}$.

Proof: By Lemma 4, J and $\tilde{J}$ are compatible and it can be shown by a straightforward calculation that the equality (being a bi-Hamiltonian system),

$$\tilde{J}\nabla\tilde{H} = J\nabla H, \tag{29}$$

or

$$\tilde{\mu}\, \nabla\, \tilde{\Psi} \times \nabla \tilde{H} = \mu\, \nabla\, \Psi \times \nabla\, H \tag{30}$$

is guaranteed by (28). Hence the system

$$\frac{dx_1}{dt} = \mu \partial_3 \Psi\, \partial_2 H - \partial_2 \Psi\, \partial_3 H,$$

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$$\frac{dx_2}{dt} = -\mu\partial_3\Psi\partial_1 H + \partial_1\Psi\partial_3 H,$$

$$\frac{dx_3}{dt} = \mu\partial_2\Psi\partial_1 H - \partial_1\Psi\partial_2 H, \tag{31}$$

is bi-Hamiltonian. □

Remark 3: The Hamiltonian function H is a conserved quantity of the system. It is clear from the expression (31) that the function Ψ is another conserved quantity of the system. Hence for a given Hamiltonian system there is a duality between H and Ψ. Such a duality arises naturally because a simple solution of the equation (28) is $\tilde{\Psi}=H$, $\tilde{H}=\Psi$ and $\tilde{\mu}=-\mu$. Thus we have a hierarchy of Hamiltonians that start with a Casimir of the second structure and terminates with a Casimir of the first structure. Such systems are equivalent to the quasi-bi-Hamiltonian systems of lower dimension with nondegenerate Poisson structures (see Ref. 5, pp. 185–220).

Remark 4: Using Lemma 5 we can construct infinitely many compatible Hamiltonian representations by choosing functions Φ, g, h satisfying (28). If we fix functions Φ and g, then Eq. (28) became linear first order partial differential equations for h. For instance, taking $g=\Psi H$ and $\tilde{\mu}=-\mu$, which fixes Φ, we obtain $h=\ln H$. Thus we a obtain second Hamiltonian representation with $\tilde{J}$ given by $\tilde{\Psi}=\Psi H$ and $\tilde{H}=\ln H$.

IV. EXAMPLES

Let us give examples of Hamiltonian systems. For each Hamiltonian system we give the Hamiltonian H and functions Ψ and μ in terms of which the corresponding Poisson structure may be written, using (4). Functions H and Ψ are first integrals of the system so one can use them to reduce the system to a first order ordinary differential equation. We give the reduced equation for the examples. We also give irregular points for the Poisson structures. For all examples except Example 7 the form of the Poisson structure (4) is preserved in a neighborhood of irregular points (function Ψ and μ are well defined). For Example 7 the form of the Poisson structure (4) is not preserved; the function Ψ is not defined in a neighborhood of irregular points but the Hamiltonian function is also not defined at the irregular points. Hence this system does not have a Hamiltonian formulation in the neighborhood of such points. Examples 6–12 satisfy the special case given in Proposition 1 and Proposition 2. Please see Ref. 1 for the examples and related references.

Example 6: For the Euler system considered in Example 1 we gave a Poisson structure in terms of functions Ψ,μ and the Hamiltonian. The reduced equations are

$$x_1 = \left(C_1 + \frac{I_1(I_3-I_2)}{I_3(I_2-I_1)}x_3^2\right)^{1/2},$$

$$x_2 = \left(C_2 + \frac{I_2(I_3-I_1)}{I_3(I_1-I_2)}x_3^2\right)^{1/2}, \tag{32}$$

$$\dot{x}_3 = \left(C_1 + \frac{I_1(I_3-I_2)}{I_3(I_2-I_1)}x_3^2\right)^{1/2}\left(C_2 + \frac{I_2(I_3-I_1)}{I_3(I_1-I_2)}x_3^2\right)^{1/2}.$$

The Poisson structure is given by (16). It has an irregular point $p=(0,0,0)$ (the origin).

Example 7: The Lotka–Voltera system considered in Example 2 has the matrix J given by $\Psi=-\ln x_1 - b\ln x_2 + c\ln x_3$, $\mu=x_1x_2x_3$ and the Hamiltonian $H=abx_1+x_2-ax_3+\nu_0\ln x_2 - \mu_0\ln x_3$.

The reduced equations can be obtained using equalities

$$-\ln x_1 - b\ln x_2 + c\ln x_3 = C_1,$$

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$$abx_1+x_2-ax_3+\nu_0\ln x_2-\mu_0\ln x_3=C_2. \tag{33}$$

The Poisson structure is given by (20). It has irregular lines given by $x_i=0$ and $x_j=0$, $i,j=1,2,3$, $j\neq i$ (coordinate lines). Both Ψ and H are not defined at these points. So, the system does not have a Hamiltonian formulation at these points.

Example 8: The Lorentz system considered in Example 3 has the matrix J given by $\Psi=\frac{1}{4}(x_3-x_1^2)$, $\mu=1$ and the Hamiltonian $H=x_1^2+x_3^2$.

The reduced equations are

$$\begin{aligned} x_1&=(C_1-x_3)^{1/2},\\ x_2&=(C_2-x_3^2)^{1/2},\\ \dot{x}_3&=(C_1-x_3)^{1/2}(C_2-x_3^2)^{1/2}. \end{aligned} \tag{34}$$

The Poisson structure is given by (22). It has no irregular points.

Example 9: Consider Kermac–Mackendric system,[8,10]

$$\begin{aligned} \dot{x}_1&=-rx_1x_2,\\ \dot{x}_2&=rx_1x_2-ax_2,\\ \dot{x}_3&=ax_2, \end{aligned} \tag{35}$$

where $r,a\in\mathbb{R}$ are constants.

The matrix J is given by $\Psi=x_1+x_2+x_3$, $\mu=x_1x_2$ and the Hamiltonian is $H=rx_3+a\ln x_1$.

The reduced equations are

$$\begin{aligned} x_2&=C_1+\frac{a}{r}\ln x_1-x_1,\\ x_3&=C_2-\frac{a}{r}\ln x_1,\\ \dot{x}_1&=-rx_1\left(C_1+\frac{a}{r}\ln x_1-x_1\right). \end{aligned} \tag{36}$$

The Poisson structure is given by

$$\begin{aligned} u&=x_1x_2,\\ v&=x_1x_2,\\ w&=x_1x_2. \end{aligned} \tag{37}$$

It has irregular planes $x_1=0$ and $x_2=0$ (coordinate planes).

Example 10: Consider the May–Leonard system,[8]

$$\begin{aligned} \dot{x}_1&=-x_2^{-\alpha}x_3^{-\alpha},\\ \dot{x}_2&=-x_1^{-\alpha}x_3^{-\alpha},\\ \dot{x}_3&=-x_1^{-\alpha}x_2^{-\alpha}. \end{aligned} \tag{38}$$

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The matrix J is given by $\Psi=[1/(1-\alpha)^2](x_2^{1-\alpha}-x_1^{1-\alpha})$, $\mu=1$ and the Hamiltonian is $H=x_1^{1-\alpha}-x_3^{1-\alpha}$, $\alpha<0$.

The reduced equations are

$$x_2=(C_1+x_1^{1-\alpha})^{1/(1-\alpha)},$$

$$x_3=(C_2+x_1^{1-\alpha})^{1/(1-\alpha)}, \tag{39}$$

$$\dot{x}_1=-(C_1+x_1^{1-\alpha})^{\alpha/(1-\alpha)}(C_2+x_1^{1-\alpha})^{\alpha/(1-\alpha)}.$$

The Poisson structure J is given by

$$u=0,$$

$$v=\frac{x_2^{-\alpha}}{\alpha-1}, \tag{40}$$

$$w=\frac{x_1^{-\alpha}}{\alpha-1}.$$

It has an irregular line $x_1=0$, $x_2=0$ (coordinate line).

Example 11: Consider the Maxvel–Bloch system,[8]

$$\dot{x}_1=x_2,$$

$$\dot{x}_2=x_1x_3, \tag{41}$$

$$\dot{x}_3=-x_1x_2.$$

The matrix J is given by $\Psi=-(1/2v)(x_2^2+x_3^2)$, $\mu=1$ and the Hamiltonian is $H=\frac{1}{2}\alpha(x_2^2+x_3^2)-(1/v)(x_3+x_1^2)$, $v\neq 0$.

The reduced equations are

$$x_1=\left(C_1+\frac{\alpha v}{2}C_2-x_3\right)^{1/2},$$

$$x_2=(C_2-x_3^2)^{1/2}, \tag{42}$$

$$\dot{x}_3=-\left(C_1+\frac{\alpha v}{2}C_2-x_3\right)^{1/2}(C_2-x_3^2)^{1/2}.$$

The Poisson structure is given by

$$u=\frac{-1}{v}x_3,$$

$$v=\frac{-1}{v}x_2, \tag{43}$$

$$w=0.$$

It has an irregular line $x_2=0$, $x_3=0$ (coordinate line).

Example 12: Consider systems that are obtained from the Lorenz system,[13]

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$$\begin{aligned}\dot{x}&=\sigma(x-y),\\ \dot{y}&=-y+rx-xz,\\ \dot{z}&=-bz+xy.\end{aligned}\tag{44}$$

Following Ref. 12, for an appropriate subset of parameters by recalling we have the following.

(i) **Lorentz(1) system:**

$$\begin{aligned}\dot{x}_1&=\sigma x_2 e^{(\sigma-1)t},\\ \dot{x}_2&=x_1 e^{(1-\sigma)t}(r-x_3 e^{-2\sigma t}),\\ \dot{x}_3&=x_1 x_2 e^{(\sigma-1)t}.\end{aligned}\tag{45}$$

The matrix J is given by $\Psi=-(r/4\sigma)\,x_1^2 e^{(1-\sigma)t}-\frac{1}{4}x_2^2 e^{(\sigma-1)t}-\frac{1}{4}x_3^2 e^{(1-3\sigma)t}$, $\mu=1$ and the Hamiltonian is $H=x_1^2-2\sigma x_3$.

The reduced equations are

$$\begin{aligned}x_1&=(C_1+2\sigma x_3)^{1/2},\\ x_2&=\left(C_2-\frac{r}{\sigma}(C_1+2\sigma x_3)e^{2(1-\sigma)t}-x_3^2 e^{2(1-2\sigma)t}\right)^{1/2},\\ \dot{x}_3&=(C_1+2\sigma x_3)^{1/2}\left(C_2-\frac{r}{\sigma}(C_1+2\sigma x_3)e^{2(1-\sigma)t}-x_3^2 e^{2(1-2\sigma)t}\right)^{1/2}e^{(1-\sigma)t}.\end{aligned}\tag{46}$$

The Poisson structure is given by

$$\begin{aligned}u&=\tfrac{1}{2}x_3 e^{(1-3\sigma)t},\\ v&=\tfrac{1}{2}x_2 e^{(\sigma-1)t},\\ w&=-\frac{r}{2\sigma}x_1 e^{(1-\sigma)t}.\end{aligned}\tag{47}$$

It has an irregular point $x_1=0$, $x_2=0$, $x_3=0$ (the origin).

(ii) **Lorentz(3) system:**

$$\begin{aligned}\dot{x}_1&=\sigma x_2 e^{(\sigma-1)t},\\ \dot{x}_2&=-x_1 x_3 e^{-\sigma t},\\ \dot{x}_3&=x_1 x_2 e^{-\sigma t}.\end{aligned}\tag{48}$$

The matrix J is given by $\Psi=-\frac{1}{4}x_1^2 e^{-\sigma t}+(\sigma/2)\,x_3 e^{(\sigma-1)t}$, $\mu=1$ and the Hamiltonian is $H=x_2^2+x_3^2$.

The reduced equations are

$$\begin{aligned}x_1&=(C_1 e^{\sigma t}+2\sigma x_3 e^{(2\sigma-1)t})^{1/2},\\ x_2&=(C_2-x_3^2)^{1/2},\\ \dot{x}_3&=(C_1 e^{\sigma t}+2\sigma x_3 e^{(2\sigma-1)t})^{1/2}(C_2-x_3^2)^{1/2}e^{-\sigma t}.\end{aligned}\tag{49}$$

The Poisson structure is given by

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$$u=\tfrac{1}{2}\sigma e^{(\sigma-1)t},$$
$$v=0, \tag{50}$$
$$w=-\tfrac{1}{2}x_1 e^{-\sigma t}.$$

It has no irregular points.

(iii) **Lorentz(5) system:**

$$\dot{x}_1=x_2,$$
$$\dot{x}_2=rx_1-x_1x_3e^{-t}, \tag{51}$$
$$\dot{x}_3=x_1x_2e^{-t}.$$

The matrix J is given by $\Psi=\frac{1}{4}x_1^2e^{-t}-\frac{1}{2}x_3$, $\mu=1$ and the Hamiltonian is $H=-rx_1^2+x_2^2+x_3^2$.

The reduced equations are

$$x_1=(C_1e^t+2x_3e^t)^{1/2},$$
$$x_2=(C_2+rC_1e^t+2rx_3e^t-x_3^2)^{1/2}, \tag{52}$$
$$\dot{x}_3=(C_1e^t+2x_3e^t)^{1/2}(C_2+rC_1e^t+2rx_3e^t-x_3^2)^{1/2}e^{-t}.$$

The Poisson structure is given by

$$u=\tfrac{1}{2},$$
$$v=0, \tag{53}$$
$$w=-\tfrac{1}{2}x_1e^{-t}0.$$

It has no irregular points.

Example 13: Consider systems that are obtained from the Rabinovich system,[14]

$$\dot{x}=-\nu_1x+hy+yz,$$
$$\dot{y}=hx-\nu_2y-xz, \tag{54}$$
$$\dot{z}=-\nu_3z+xy.$$

Following Ref. 12, for an appropriate subset of parameters by recalling we have the following.

(i) **Rabinovich (1) system:**

$$\dot{x}_1=hx_2+x_2x_3e^{-2\nu t},$$
$$\dot{x}_2=hx_1-x_1x_3e^{-2\nu t}, \tag{55}$$
$$\dot{x}_3=x_1x_2.$$

The matrix J is given by $\Psi=\frac{1}{8}x_1^2-\frac{1}{8}x_2^2-\frac{1}{4}x_3^2e^{-2\nu t}$, $\mu=1$ and the Hamiltonian is $H=x_1^2+x_2^2-4hx_3$.

The reduced equations are

$$x_1=(C_1+x_3^2e^{-2\nu t}+2hx_3)^{1/2},$$
$$x_2=(C_2-x_3^2e^{-2\nu t}+2hx_3)^{1/2}, \tag{56}$$
$$\dot{x}_3=(C_1+x_3^2e^{-2\nu t}+2hx_3)^{1/2}(C_2-x_3^2e^{-2\nu t}+2hx_3)^{1/2}.$$

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The Poisson structure is given by

$$u=\tfrac{1}{2}x_3e^{-2\nu t},$$
$$v=\tfrac{1}{4}x_2, \tag{57}$$
$$w=-\tfrac{1}{4}x_1.$$

It has an irregular point $x_1=0$, $x_2=0$, $x_3=0$ (the origin).

(ii) **Rabinovich (2) system:**

$$\dot{x}_1=hx_2+x_2x_3e^{-\nu t},$$
$$\dot{x}_2=hx_1-x_1x_3e^{-\nu t}, \tag{58}$$
$$\dot{x}_3=x_1x_2e^{-\nu t}.$$

The matrix J is given by $\Psi=\frac{1}{8}x_1^2e^{-\nu t}+\frac{1}{8}x_2^2e^{-\nu t}-\frac{1}{2}hx_3$, $\mu=1$ and the Hamiltonian is $H=x_1^2-x_2^2-2x_3^2$.

The reduced equations are

$$x_1=(C_1e^{\nu t}+C_2+x_3^2+2hx_3e^{\nu t})^{1/2},$$
$$x_2=(C_1e^{\nu t}-C_2-x_3^2+2hx_3e^{\nu t})^{1/2}, \tag{59}$$
$$\dot{x}_3=(C_1e^{\nu t}+C_2+x_3^2+2hx_3e^{\nu t})^{1/2}(C_1e^{\nu t}-C_2-x_3^2+2hx_3e^{\nu t})^{1/2}e^{-\nu t}.$$

The Poisson structure is given by

$$u=-\tfrac{1}{2}h,$$
$$v=\tfrac{1}{4}x_2e^{-\nu t}, \tag{60}$$
$$w=\tfrac{1}{4}x_1e^{-\nu t}.$$

It has no irregular points.

(iii) **Rabinovich (3) system:**

$$\dot{x}_1=x_2x_3e^{\nu_3 t},$$
$$\dot{x}_2=-x_1x_3e^{-\nu_3 t}, \tag{61}$$
$$\dot{x}_3=x_1x_2e^{(\nu_3-2\nu)t}.$$

The matrix J is given by $\Psi=\frac{1}{4}x_2^2e^{(\nu_3-2\nu)t}+\frac{1}{4}x_3^2e^{-\nu_3 t}$, $\mu=1$ and the Hamiltonian is $H=x_1^2+x_2^2$.

The reduced equations are

$$x_1=(C_1-x_2^2)^{1/2},$$
$$x_3=(C_2e^{-\nu_3 t}-x_3^2e^{-2(\nu-\nu_3)t})^{1/2}, \tag{62}$$
$$\dot{x}_2=(C_1-x_2^2)^{1/2}(C_2e^{-\nu_3 t}-x_3^2e^{-2(\nu-\nu_3)t})^{1/2}e^{(\nu_3-2\nu)t}.$$

The Poisson structure is given by

$$u=\tfrac{1}{2}x_3e^{-\nu_3 t},$$
$$v=\tfrac{1}{2}x_2e^{(\nu_3-2\nu)t}, \tag{63}$$
$$w=0.$$

It has an irregular line $x_2=0$, $x_3=0$ (coordinate line).

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(iv) **Rabinovich (4) system:**

$$\dot{x}_1 = hx_2 e^{\nu_1 t} + x_2 x_3 e^{\nu_1 t},$$
$$\dot{x}_2 = hx_1 e^{-\nu_1 t} - x_1 x_3 e^{-\nu_1 t}, \tag{64}$$
$$\dot{x}_3 = x_1 x_2 e^{-\nu_1 t}.$$

The matrix J is given by $\Psi = -\frac{1}{4}x_1^2 e^{-\nu t} - \frac{1}{4}x_2^2 e^{\nu_1 t} + hx_3 e^{\nu_1 t}$, $\mu = 1$ and the Hamiltonian is $H = x_2^2 + (h - x_3)^2$.

The reduced equations are

$$x_1 = (C_1 e^{\nu t} - (C_2 - (h + x_3))e^{(\nu_1 + \nu)t})^{1/2},$$
$$x_2 = (C_2 - (h - x_3)^2)^{1/2}, \tag{65}$$
$$\dot{x}_3 = (C_1 e^{\nu t} - (C_2 - (h + x_3))e^{(\nu_1 + \nu)t})^{1/2}(C_2 - (h - x_3)^2)^{1/2} e^{-\nu_1 t}.$$

The Poisson structure is given by

$$u = he^{\nu_1 t},$$
$$v = -\tfrac{1}{2}x_2 e^{\nu_1 t}, \tag{66}$$
$$w = -\tfrac{1}{2}x_1 e^{-\nu t}.$$

It has no irregular points.

(v) **Rabinovich (5) system:**

$$\dot{x}_1 = hx_2 e^{-\nu_2 t} + x_2 x_3 e^{-\nu_2 t},$$
$$\dot{x}_2 = hx_1 e^{\nu_2 t} - x_1 x_3 e^{\nu_2 t}, \tag{67}$$
$$\dot{x}_3 = x_1 x_2 e^{-\nu_2 t}.$$

The matrix J is given by $\Psi = \frac{1}{4}x_1^2 e^{\nu_2 t} + \frac{1}{4}x_2^2 e^{-\nu_2 t} - hx_3 e^{\nu_2 t}$, $\mu = 1$ and the Hamiltonian is $H = x_1^2 - (h + x_3)^2$.

The reduced equations are

$$x_1 = (C_1 + (h + x_3)^2)^{1/2},$$
$$x_2 = (C_2 e^{\nu_2 t} - (C_1 + (h - x_3))e^{2\nu_2 t})^{1/2}, \tag{68}$$
$$\dot{x}_3 = (C_1 + (h + x_3)^2)^{1/2}(C_2 - (C_1 + (h - x_3))e^{2\nu_2 t})^{1/2} e^{-\nu_2 t}.$$

The Poisson structure is given by

$$u = -he^{\nu_2 t},$$
$$v = \tfrac{1}{2}x_2 e^{-\nu_2 t}, \tag{69}$$
$$w = \tfrac{1}{2}x_1 e^{\nu_2 t}.$$

It has no irregular points.

(vi) **Rabinovich (6) system:**

$$\dot{x}_1 = x_2 x_3 e^{(\nu_1 - 2\nu_3)t},$$
$$\dot{x}_2 = -x_1 x_3 e^{-\nu_1 t}, \tag{70}$$
$$\dot{x}_3 = x_1 x_2 e^{-\nu_1 t}.$$

The matrix J is given by $\Psi = -\frac{1}{4}x_1^2 e^{-\nu_1 t} - \frac{1}{4}x_2^2 e^{(\nu_1 - 2\nu_2)t}$, $\mu = 1$ and the Hamiltonian is $H = x_2^2 + x_3^2$.

The reduced equations are

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$$x_1=(C_1e^{\nu_1 t}+x_2^2e^{2(\nu_1-\nu_2)t})^{1/2},$$

$$x_3=(C_2-x_2^2)^{1/2}, \tag{71}$$

$$\dot{x}_2=-(C_1e^{\nu_1 t}+x_2^2e^{2(\nu_1-\nu_2)t})^{1/2}(C_2-x_2^2)^{1/2}e^{-\nu_1 t}.$$

The Poisson structure is given by

$$u=0,$$

$$v=-\tfrac{1}{2}x_2e^{(\nu_1-2\nu_2)t}, \tag{72}$$

$$w=-\tfrac{1}{2}x_1e^{-\nu_1 t}.$$

It has an irregular line $x_1=0$, $x_2=0$ (coordinate line).

(vii) **Rabinovich (7) system:**

$$\dot{x}_1=x_2x_3e^{-\nu_2 t},$$

$$\dot{x}_2=-x_1x_3e^{(\nu_2-2\nu_3)t}, \tag{73}$$

$$\dot{x}_3=x_1x_2e^{-\nu_2 t}.$$

The matrix J is given by $\Psi=\frac{1}{4}x_1^2e^{(\nu_2-2\nu_3)t}+\frac{1}{4}x_2^2e^{-\nu_2 t}$, $\mu=1$ and the Hamiltonian is $H=x_1^2-x_3^2$.

The reduced equations are

$$x_2=(C_1e^{\nu_2 t}-x_1^2e^{2(\nu_2-\nu_3)t})^{1/2},$$

$$x_3=(C_2+x_1^2)^{1/2}, \tag{74}$$

$$\dot{x}_1=(C_1e^{\nu_2 t}-x_1^2e^{2(\nu_2-\nu_3)t})^{1/2}(C_2+x_1^2)^{1/2}e^{-\nu_2 t}.$$

The Poisson structure is given by

$$u=0,$$

$$v=\tfrac{1}{2}x_2e^{\nu_2 t}, \tag{75}$$

$$w=\tfrac{1}{2}x_1e^{\nu_2-2\nu_3 t}.$$

It has an irregular line $x_2=0$, $x_3=0$ (coordinate line).

Example 14: Consider systems that are obtained from the RTW system,[14]

$$\dot{x}=\gamma x+\delta y+z-2y^2,$$

$$\dot{y}=\gamma y-\delta x+2xy, \tag{76}$$

$$\dot{z}=-2z(x+1),$$

for an appropriate subset of parameters by recalling. Following Ref 12 we have the following.

(i) **RTW(1) system:**

$$\dot{x}_1=\delta x_2+x_3e^{-2t}-2x_2^2,$$

$$\dot{x}_2=-\delta x_1+2x_1x_2, \tag{77}$$

$$\dot{x}_3=-x_1x_3,$$

where δ is an arbitrary constant. The matrix J is given by $\Psi=\frac{1}{2}(x_1^2-x_2^2+x_3e^{-t})$, $\mu=1$ and the Hamiltonian is $H=x_3(2x_2-\delta)$.

The reduced equations are

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$$x_1=\left(C_1-x_3e^{-t}+\left(\frac{C_2+\delta x_3}{2x_3}\right)^2\right)^{1/2},$$
$$x_2=\frac{C_2+\delta x_3}{2x_3},$$
$$\dot{x}_3=-\left(C_1-x_3e^{-t}+\left(\frac{C_2+\delta x_3}{2x_3}\right)^2\right)^{1/2}x_3. \tag{78}$$

The Poisson structure is given by

$$u=\frac{1}{2}e^{-2t},$$
$$v=x_2, \tag{79}$$
$$w=x_1.$$

It has no irregular points.

(ii) **RTW(2) system:**

$$\dot{x}_1=\delta x_2+x_3e^{-t}-2x_2^2e^{-t},$$
$$\dot{x}_2=-\delta x_1+2x_1x_2e^{-t}, \tag{80}$$
$$\dot{x}_3=-x_1x_3e^{-t},$$

where δ is an arbitrary constant. The matrix J is given by $\Psi=-(\delta/2)(x_1^2+x_2^2)-x_3x_2e^{-t}$, $\mu=1$ and the Hamiltonian is $H=x_1^2+x_2^2+x_3$.

The reduced equations are

$$x_1=\left(C_2-x_3-\left(C_1e^t-\frac{\delta}{2}C_2+\frac{\delta}{2}x_3\right)^2\right)^{1/2},$$
$$x_2=C_1e^t-\frac{\delta}{2}C_2+\frac{\delta}{2}x_3, \tag{81}$$
$$\dot{x}_3=\left(C_2-x_3-\left(C_1e^t-\frac{\delta}{2}C_2+\frac{\delta}{2}x_3\right)^2\right)^{1/2}x_3e^{-t}.$$

The Poisson structure is given by

$$u=-x_2e^{-t},$$
$$v=-\delta x_2-x_3e^{-t}, \tag{82}$$
$$w=-\delta x_1.$$

It has an irregular point $x_1=0$, $x_2=0$, $x_3=0$ (the origin).

(iii) **RTW(3) system:**

$$\dot{x}_1=(x_3-2x_2)e^{-t},$$
$$\dot{x}_2=2x_1x_2e^{-t}, \tag{83}$$
$$\dot{x}_3=-2x_1x_3e^{-t}.$$

The matrix J is given by $\Psi=(x_1^2-x_2^2+x_3)e^{-t}$, $\mu=1$ and the Hamiltonian is $H=x_2x_3$.

The reduced equations are

$$x_1=\left(C_1e^t-x_3-\frac{C_2^2}{x_3^2}\right)^{1/2},$$
$$x_2=\frac{C_2}{x_3}, \tag{84}$$
$$\dot{x}_3=-2\left(C_1e^t-x_3-\frac{C_2^2}{x_3^2}\right)^{1/2}x_3e^{-t}.$$

The Poisson structure is given by

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$$u=e^{-t},\quad v=2x_2e^{-t},\quad w=2x_1e^{-t}. \tag{85}$$

It has no irregular points.

(iv) **RTW(4) system:**

$$\dot{x}_1=x_3e^{-(\gamma+2)t}-2x_2^2e^{\gamma t},\quad \dot{x}_2=2x_1x_2e^{\gamma t},\quad \dot{x}_3=-2x_1x_3e^{\gamma t}, \tag{86}$$

where γ is an arbitrary constant. The matrix J is given by $\Psi=(x_1^2-x_2^2)e^{\gamma t}+x_3e^{-(\gamma+2)t}$, $\mu=1$ and the Hamiltonian is $H=x_2x_3$.

The reduced equations are

$$x_1=\left(C_1e^{-\gamma t}-x_3e^{-2(\gamma+1)t}+\frac{C_2^2}{x_3^2}\right)^{1/2},\quad x_2=\frac{C_2}{x_3},\quad \dot{x}_3=-2\left(C_1e^{-\gamma t}-x_3e^{-2(\gamma+1)t}+\frac{C_2^2}{x_3^2}\right)^{1/2}x_3e^{\gamma t}. \tag{87}$$

The Poisson structure is given by

$$u=e^{-(2+\gamma)t},\quad v=2x_2e^{\gamma t},\quad w=2x_1e^{\gamma t}. \tag{88}$$

It has no irregular points.

(v) **RTW(5) system:**

$$\dot{x}_1=\delta x_2+x_3-2x_2^2e^{-2t},\quad \dot{x}_2=-\delta x_1+2x_1x_2e^{-2t},\quad \dot{x}_3=-2x_1x_3e^{-2t}, \tag{89}$$

where δ is a nonvanishing constant. The matrix J is given by $\Psi=(\delta e^{-2t}/2)(x_1^2-x_2^2)+(\delta/2)x_3$, $\mu=1$ and the Hamiltonian is $H=x_1^2+x_2^2+(2/\delta)x_2x_3$.

The reduced equations are

$$x_1=\left(C_1e^{2t}+x_2^2+e^{2t}\frac{C_2-C_1e^{2t}-2x_2^2}{\frac{\delta}{2}x_2+e^{2t}}\right)^{1/2},\quad x_3=\frac{C_2-C_1e^{2t}-2x_2^2}{\frac{\delta}{2}x_2+e^{2t}}, \tag{90}$$

$$\dot{x}_2=-\delta\left(C_1e^{2t}+x_2^2+e^{2t}\frac{C_2-C_1e^{2t}-2x_2^2}{\frac{\delta}{2}x_2+e^{2t}}\right)^{1/2}+2\left(C_1e^{2t}+x_2^2+e^{2t}\frac{C_2-C_1e^{2t}-2x_2^2}{\frac{\delta}{2}x_2+e^{2t}}\right)^{1/2}x_2e^{-2t}.$$

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TABLE I. Examples of Hamiltonian systems given in the text. In each example we give a Hamiltonian H and a Poisson structure J [J is given in terms of μ, Ψ by Eq. (4)].

	Poisson matrix		Hamiltonian
System	Ψ	μ	
Euler	$-\frac{1}{2}(x_1^2+x_2^2+x_3^2)$	1	$\frac{x_1^2}{2I_1}+\frac{x_2^2}{2I_2}+\frac{x_3^2}{2I_3}$
Lotka–Voltera	$\ln\frac{x_3^c x_2^{bc}}{x_1}$	$x_1x_2x_3$	$a(bx_1-x_3)+x_2+\ln\frac{x_2^\nu}{x_3^\mu}$
Lorenz	$\frac{1}{4}(x_3-x_1^2)$	1	$x_2^2+x_3^2$
Kermac–Mackendric	$x_1+x_2+x_3$	x_1x_2	$a\ln x_1+rx_3$
May–Leonard	$\frac{1}{(1-\alpha)^2}(x_2^{1-\alpha}-x_1^{1-\alpha})$	1	$x_1^{1-\alpha}-x_3^{1-\alpha}$
Maxvel–Bloch	$-\frac{1}{2v}(x_2^2+x_3^2)$	1	$\frac{1}{2}\alpha(x_2^2+x_3^2)-\frac{1}{v}(x_3+x_1^2)$
Lor.(1)	$-\left(\frac{r}{\sigma}x_1^2+x_2^2\right)\frac{e^{(\sigma-1)t}}{4}-x_3^2\frac{e^{(1-3\sigma)t}}{4}$	1	$x_1^2-2\sigma x_3$
Lor.(3)	$-\frac{1}{4}x_1^2e^{-\sigma t}+\frac{\sigma}{2}x_3e^{(\sigma-1)t}$	1	$x_2^2+x_3^2$
Lor.(5)	$\frac{1}{4}x_1^2e^{-t}-\frac{1}{2}x_3$	1	$-rx_1^2+x_2^2+x_3^2$
Rab.(1)	$\frac{1}{8}x_1^2-\frac{1}{8}x_2^2-\frac{1}{4}x_3^2e^{-2\nu t}$	1	$x_1^2+x_2^2-4hx_3$
Rab.(2)	$\frac{1}{8}x_1^2e^{-\nu t}+\frac{1}{8}x_2^2e^{-\nu t}-\frac{1}{2}hx_3$	1	$x_1^2-x_2^2-2x_3^2$
Rab.(3)	$\frac{1}{4}x_2^2e^{(\nu_3-2\nu)t}+\frac{1}{4}x_3^2e^{-\nu_3 t}$	1	$x_1^2+x_2^2$
Rab.(4)	$-\frac{1}{4}x_1^2e^{-\nu t}-\frac{1}{4}x_2^2e^{\nu_1 t}+hx_3e^{\nu_1 t}$	1	$x_2^2+(h-x_3)^2$
Rab.(5)	$\frac{1}{4}x_1^2e^{\nu_2 t}+\frac{1}{4}x_2^2e^{-\nu_2 t}-hx_3e^{\nu_2 t}$	1	$x_1^2-(h+x_3)^2$
Rab.(6)	$-\frac{1}{4}x_1^2e^{-\nu_1 t}-\frac{1}{4}x_2^2e^{(\nu_1-2\nu_2)t}$	1	$x_2^2+x_3^2$
Rab.(7)	$\frac{1}{4}x_1^2e^{(\nu_2-2\nu_3)t}+\frac{1}{4}x_2^2e^{-\nu_2 t}$	1	$x_1^2-x_3^2$
RTW.(1)	$\frac{1}{2}(x_1^2-x_2^2+x_3e^{-t})$	1	$x_3(2x_2-\delta)$
RTW.(2)	$-\frac{\delta}{2}x_1^2-\left(\frac{\delta}{2}x_2^2+x_3x_2\right)e^{-t}$	1	$x_1^2+x_2^2+x_3$
RTW.(3)	$(x_1^2-x_2^2+x_3)e^{-t}$	1	x_2x_3
RTW.(4)	$(x_1^2-x_2^2)e^{\gamma t}+x_3e^{-(\gamma+2)t}$	1	x_2x_3
RTW.(5)	$\frac{\delta e^{-2t}}{2}(x_1^2-x_2^2)+\frac{\delta}{2}x_3$	1	$x_1^2+x_2^2+\frac{2}{\delta}x_2x_3$

The Poisson structure is given by

$$u=e^{-(2+\gamma)t},\qquad v=2x_2e^{\gamma t},\qquad w=2x_1e^{\gamma t}. \tag{91}$$

It has no irregular points.

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V. CONCLUSION

We considered the Jacobi equation for the case $N=3$. We have found the most general Poisson structure J in the neighborhood of regular points. This form is quite suitable for the study of the multi-Hamiltonian structure of the system. We found all possible compatible Poisson structures and corresponding bi-Hamiltonian systems. We studied our solution in the neighborhood of the irregular points of the Poisson structure and showed that it keeps its form. As an application of our results we gave several examples which were reported earlier[8–15] as bi-Hamiltonian systems. In these examples we give the Casimirs, components of the Poisson matrix, the reduced equations and irregular points. Among all examples that we observed, only the Lotka–Voltera system has a special position. Our solution is not valid in the neighborhood of irregular points for this system. On the other hand the Hamiltonian function is not defined at such points as well. Hence the Lotka–Voltera equation does not have the Hamiltonian formulation in the neighborhood of such points.

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