Research paper

Nonlocal hydrodynamic type of equations

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ABSTRACT

We show that the integrable equations of hydrodynamic type admit nonlocal reductions. We first construct such reductions for a general Lax equation and then give several examples. The reduced nonlocal equations are of hydrodynamic type and integrable. They admit Lax representations and hence possess infinitely many conserved quantities.

1. Introduction

Let us consider the following system of evolution equations in $(1 + 1)$-dimension,

$$q_t^i = F^i(q^k, r^k, q_x^k, r_x^k, q_{xx}^k, r_{xx}^k, \ldots), \tag{1}$$

$$r_t^i = G^i(q^k, r^k, q_x^k, r_x^k, q_{xx}^k, r_{xx}^k, \ldots), \tag{2}$$

for all $i, k = 1, 2, \ldots, N$. Here F^i and G^i, $(i = 1, 2, \ldots, N)$ are functions of the dynamical variables $q^i(x, t)$, $r^i(x, t)$, and their partial derivatives with respect to x. If the above system has a Lax pair then it is an integrable system. There are local and nonlocal reductions of the coupled system (1) and (2). The local reductions are given by

$$r^i(x, t) = \rho q^i(x, t), \tag{3}$$

and

$$r^i(x, t) = \rho \bar{q}^i(x, t), \tag{4}$$

where ρ is a real constant and a bar over a letter denotes complex conjugation. By these reductions the systems of Eqs. (1) and (2) reduce to one system for q^i's

$$q_t^i = \tilde{F}^i(q^j, q_x^j, q_{xx}^j, \ldots), \quad i, j = 1, 2, \ldots, N, \tag{5}$$

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so that the system (2) consistently reduces to the above system (5). Here corresponding to (3) we have $\tilde{F} = F|_{r=\rho q}$ and $\tilde{F} = F|_{r=\rho \bar{q}}$ corresponding to (4). There are also nonlocal reductions introduced by Ablowitz and Musslimani [1–3] given as

$$r^i(x, t) = \rho q^i(\varepsilon_1 x, \varepsilon_2 t), \tag{6}$$

and

$$r^i(x, t) = \rho \bar{q}^i(\varepsilon_1 x, \varepsilon_2 t), \tag{7}$$

for $i = 1, 2, \ldots, N$. Here ρ is a real constant and $\varepsilon_1^2 = \varepsilon_2^2 = 1$. If $\varepsilon_1 = \varepsilon_2 = 1$ the above reductions turn to be local reductions. When $(\varepsilon_1, \varepsilon_2) = (1, -1), (-1, 1), (-1, -1)$, we reduce the system (1) and (2) by these nonlocal reductions to nonlocal time reflection symmetric (T-symmetric), space reflection symmetric (S-symmetric), or space-time reflection symmetric (ST-symmetric) differential equations.

The nonlocal reductions of systems of integrable equations were first consistently applied to the nonlinear Schrödinger system, which led to the nonlocal nonlinear Schrödinger equation (nNLS) [1–3]. It has been later shown that there are many other systems where the consistent nonlocal reductions are possible [4–29]. For a recent review of this subject see [27]. See also [30] for the discussion of superposition of nonlocal integrable equations and [31] for the origin of nonlocal reductions.

One way of obtaining a system of integrable equations, in Gelfand-Dikii formalism [32], is to construct a Lax operator on some Lie algebra. The examples of such algebras are the matrix algebra, algebra of differential operators, and algebra of Laurent series (see [33–35] and references there in). In most cases the Lax operator is polynomial. It is a polynomial of the spectral parameter in the matrix algebra, the operator D_x in the algebra of differential operators, and an auxiliary variable p (momentum) in the case of the algebra of Laurent series.

When we write the Lax equation on the algebra of Laurent series we derive equations of the hydrodynamic type, that is a system of first order quasilinear partial differential equations

$$u_{nt} = \sum_{m=1}^{k} h_{mn}(u) u_{mx}, \qquad n = 1, 2, \ldots, k, \tag{8}$$

where $u = (u_1, \ldots, u_k)$. The derived systems are integrable. This means they admit recursion operators and multi-Hamiltonian representation (see [33,36–38]).

For a hydrodynamic type system we also have local reductions (3), (4), and nonlocal reductions (6), (7) consistent for $\rho^2 = 1$. Local reductions for the hydrodynamic type systems were considered in [36]. In this work we address to the problem of nonlocal reductions for the Lax equations on the algebra of Laurent series.

The layout of the paper is as follows. In Section 2 we give a short review on the hydrodynamic type of (dispersionless) equations and their Lax representations. In Section 3 we give nonlocal reductions for real and complex valued fields for $N = 2$. We obtain some explicit examples of nonlocal hydrodynamic type of equations with some conserved quantities. In Section 4 we discuss the nonlocal reductions in general and give examples for higher values of N.

2. Lax equations

In this section we describe the algebra of Laurent series, introduce necessary definitions to write a Lax equation, and give some examples. In later sections we give nonlocal reductions of these equations.

We consider an algebra $\mathcal{A}$, given by

$$\mathcal{A} = \left\{ \alpha = \sum_{-\infty}^{\infty} \alpha_n(x, t) p^n : \alpha_n(x, t) \in Y \right\}, \tag{9}$$

where Y is either $C^\infty(S^1)$-the space of periodic functions or $S(\mathbb{R})$-the space of smooth asymptotically decreasing functions on $\mathbb{R}$. A Poisson bracket on $\mathcal{A}$, is given by

$$\{f, g\}_k = p^k \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \right), \qquad f, g \in \mathcal{A}, \tag{10}$$

where $k \in \mathbb{Z}$.

On the algebra $\mathcal{A}$ we can define a trace functional

$$\mathrm{tr}_k \alpha = \int_I \mathrm{Res}_k \alpha \, dx, \qquad \alpha \in \mathcal{A}, \tag{11}$$

where $\mathrm{Res}_k \alpha = \alpha_{k-1}(x, t)$. The set I is either S^1 for the space of periodic functions or $\mathbb{R}$ for the space of asymptotically decaying functions. Using the trace functional we define a non-degenerate, ad-invariant, symmetric pairing

$$(\alpha, \beta) = \mathrm{tr}_k \alpha \cdot \beta, \qquad \alpha, \beta \in \mathcal{A}. \tag{12}$$

The algebra $\mathcal{A}$ admits a decomposition into sub-algebras, closed with respect to the Poisson bracket,

$$\mathcal{A} = \mathcal{A}_{\geq -k+1} \oplus \mathcal{A}_{< -k+1}, \tag{13}$$

where

$$\mathcal{A}_{\geq -k+1} = \left\{ \alpha_{\geq -k+1} = \sum_{-k+1}^{\infty} \alpha_n(x,t)p^n : \alpha_n(x,t) \in Y \right\} \tag{14}$$

and

$$\mathcal{A}_{< -k+1} = \left\{ \alpha_{< -k+1} = \sum_{-\infty}^{-k} \alpha_n(x,t)p^n : \alpha_n(x,t) \in Y \right\}. \tag{15}$$

So, we can define r-matrix mapping on $\mathcal{A}$ (see [35] and references therein)

$$R_k = \frac{1}{2} \left(\Pi_{\geq -k+1} - \Pi_{< -k+1} \right),$$

where $\Pi_{\geq -k+1}$ and $\Pi_{< -k+1}$ are projections on the sub-algebras $\mathcal{A}_{\geq -k+1}$ and $\mathcal{A}_{< -k+1}$ respectively.

Now for a Lax operator

$$L = p^{N-1} + \sum_{i=-1}^{N-2} p^i S_i(x,t), \qquad N \in \mathbb{N}, \tag{16}$$

we can consider a hierarchy of the Lax equations (see [36])

$$\frac{\partial L}{\partial t_n} = \left\{ R_k(L^{\frac{m}{N-1}+n}), L \right\}_k = \left\{ L^{\frac{m}{N-1}+n}_{\geq -k+1}, L \right\}_k, \qquad n = 1, 2, \ldots, \tag{17}$$

where $m = 0, 1, 2, \ldots, (N-2)$ is fixed. This is an integrable hierarchy that is the Lax equations admit multi-Hamiltonian representation (see [33]). In particular the conserved quantities for the hierarchy are given by

$$\mathcal{Q}_n = \mathrm{tr}_k L^{\frac{m}{N-1}+n}, \qquad n = 1, 2, \ldots. \tag{18}$$

We note that to consider nonlocal reductions it is more convenient to write the Eq. (17) in terms of zeros of the Lax operator. That is we set

$$L = \frac{1}{p} \prod_{j=1}^{N} (p - u_j). \tag{19}$$

Let us give some examples of the Lax equations and their conserved quantities. First, we consider examples of Lax equations corresponding to the Lax operator

$$L = \frac{1}{p}(p - u)(p - v). \tag{20}$$

Example 1. We take the algebra $\mathcal{A}$ with $k = 0$ and consider the Lax Eq. (17) with a Lax operator (20). For $n = 2$, we obtain the shallow water waves system

$$\frac{1}{2} u_{t_2} = (u+v)u_x + uv_x, \frac{1}{2} v_{t_2} = (u+v)v_x + vu_x. \tag{21}$$

Taking $n = 3, 4, \ldots$, we can have generalized symmetries of the shallow water waves system. For instance, the first two symmetries are

$$\frac{1}{3} u_{t_3} = (u^2 + 4uv + v^2)u_x + (2u^2 + 2uv)v_x, \frac{1}{3} v_{t_3} = (2v^2 + 2uv)u_x + (v^2 + 4uv + u^2)v_x, \tag{22}$$

and

$$\begin{aligned} \frac{1}{4} u_{t_4} &= \left(u^3 + 9u^2v + 9uv^2 + v^3 \right)u_x + \left(3u^3 + 9u^2v + 3uv^2 \right)v_x, \\ \frac{1}{4} v_{t_4} &= \left(3v^3 + 9v^2u + 3vu^2 \right)u_x + \left(v^3 + 9v^2u + 9vu^2 + u^3 \right)v_x. \end{aligned} \tag{23}$$

The above systems admit infinitely many conserved quantities. The first three of the conserved quantities are

$$\mathcal{Q}_2 = \int_I (u^2v + uv^2)\, dx, \tag{24}$$

$$\mathcal{Q}_3 = \int_I (u^3v + 3u^2v^2 + uv^3)\, dx, \tag{25}$$

$$\mathcal{Q}_4 = \int_I (u^4v + 6u^3v^2 + 6u^2v^3 + 6uv^4)\, dx. \tag{26}$$

Example 2. We take the algebra $\mathcal{A}$ with $k = 1$ and consider the Lax Eq. (17) with a Lax operator (20). For $n = 1$ we obtain the Toda system

$$u_{t_1} = uv_x, \quad v_{t_1} = vu_x. \tag{27}$$

Taking $n = 2, 3, \ldots$, we can have generalized symmetries of the Toda system. The Toda system admits infinitely many conserved quantities. The first three of the conserved quantities are

$$\mathcal{Q}_1 = \int_I (u + v) \, dx, \tag{28}$$

$$\mathcal{Q}_2 = \int_I (u^2 + 4uv + v^2) \, dx, \tag{29}$$

$$\mathcal{Q}_3 = \int_I (u^3 + 9u^2 v + 9uv^2 + v^3) \, dx. \tag{30}$$

Let us give examples of the Lax equation with more general Lax operator.

Example 3. We take the algebra $\mathcal{A}$ with $k = 0$ and consider the Lax Eq. (17) with a Lax operator

$$L = \frac{1}{p}(p - u)(p - v)(p - w), \tag{31}$$

and $m = 1$. For $n = 1$ we obtain the following system

$$\begin{aligned}
\frac{8}{3} u_{t_1} &= \left(-u^2 + v^2 + w^2 + 4uv + 4uw + 6vw\right)u_x + \left(2u^2 + 2uv + 6uw\right)v_x + \left(2u^2 + 6uv + 2uw\right)w_x, \\
\frac{8}{3} v_{t_1} &= \left(2v^2 + 2uv + 6vw\right)u_x + \left(-v^2 + u^2 + w^2 + 4uv + 4vw + 6vw\right)v_x + \left(2v^2 + 6uv + 2vw\right)w_x, \\
\frac{8}{3} w_{t_1} &= \left(2w^2 + 6wv + 2uw\right)u_x + \left(2w^2 + 2wv + 6uw\right)v_x + \left(-w^2 + v^2 + u^2 + 4wv + 4uw + 6uv\right)w_x.
\end{aligned} \tag{32}$$

Taking $n = 2, 3, \ldots$, we can have generalized symmetries of the above system. The above system admits infinitely many conserved quantities. The first two of the conserved quantities are

$$\mathcal{Q}_1 = \int_I (u^2 + v^2 + w^2 - 2uv - 2uw - 2vw) \, dx, \tag{33}$$

$$\begin{aligned}
\mathcal{Q}_2 = \int_I (u^4 + v^4 + w^4 - 4(u^3 v - u^3 w - uv^3 - v^3 w - uw^3 - vw^3) + 6(u^2 v^2 + u^2 w^2 + 6v^2 w^2) \\
+ 36(u^2 vw + uv^2 w + uvw^2)) \, dx,
\end{aligned} \tag{34}$$

Example 4. For simplicity, we take the algebra $\mathcal{A}$ with $k = 0$ and consider the Lax Eq. (17) with a Lax operator

$$L = \frac{1}{p}(p - u)(p - u_1)(p - v)(p - v_1), \tag{35}$$

and $m = 0$. For $n = 1$ we obtain the following system

$$\begin{aligned}
u_{t_1} &= (u_1 v + u_1 v_1 + vv_1)u_x + (uv + uv_1)u_{1x} + (uu_1 + uv_1)v_x + (uu_1 + uv)v_{1x}, \\
u_{1t_1} &= (uv + uv_1 + vv_1)u_{1x} + (u_1 v + u_1 v_1)u_x + (uu_1 + u_1 v_1)v_x + (u_1 u + u_1 v)v_{1x}, \\
v_{t_1} &= (uu_1 + uv_1 + u_1 v_1)v_x + (u_1 v + vv_1)u_x + (uv + vv_1)u_{1x} + (uv + u_1 v)v_{1x}, \\
v_{1t_1} &= (uu_1 + uv + u_1 v)v_{1x} + (u_1 v_1 + vv_1)u_x + (uv_1 + vv_1)u_{1x} + (uv_1 + u_1 v_1)v_x.
\end{aligned} \tag{36}$$

Taking $n = 2, 3, \ldots$, we can have generalized symmetries of the above system. The above system admits infinitely many conserved quantities. The first two of the conserved quantities are

$$\mathcal{Q}_1 = \int_I uu_1 vv_1 \, dx, \tag{37}$$

$$\mathcal{Q}_2 = \int_I (u^2 u_1^2 v^2 v_1 + u^2 u_1^2 vv_1^2 + u^2 u_1 v^2 v_1^2 + uu_1^2 v^2 v_1^2) \, dx. \tag{38}$$

3. Nonlocal reductions for two component Lax operator

To study nonlocal reductions of the Lax Eq. (17), we start with the case of two component systems. That is we study the hierarchy

$$\frac{\partial L}{\partial t_n} = \left\{ L_{\geq -k+1}^n; L \right\}_k, \qquad n = 1, 2, \ldots \tag{39}$$

with the Lax operator (20).

3.1. Nonlocal reductions for real valued dependent variables

In this section we assume that our dependent variables are real valued and consider reduction of the form

$$v(x, t) = \rho u(\varepsilon_1 x, \varepsilon_2 t) = \rho u^\varepsilon. \tag{40}$$

The reduced equation is the Lax Eq. (39) with the Lax operator

$$L = \frac{1}{p}(p - u)(p - \rho u^\varepsilon). \tag{41}$$

To find admissible reductions we introduce the following change of variables

$$\tilde{x} = \varepsilon_1 x, \quad \tilde{t}_n = \varepsilon_2 t_n, \quad \tilde{p} = \rho p. \tag{42}$$

In new variables the Lax operator becomes

$$L = \frac{\rho}{\tilde{p}}(\rho \tilde{p} - u^\varepsilon)(\rho \tilde{p} - \rho u) = \rho \tilde{L}, \tag{43}$$

where $\tilde{L} = \frac{1}{\tilde{p}}(\tilde{p} - u)(\tilde{p} - \rho u^\varepsilon)$. The Lax Eq. (39) takes the form

$$\varepsilon_2 \frac{\partial}{\partial \tilde{t}_n}(\rho \tilde{L}) = (\rho \tilde{p})^k \left(\rho \frac{\partial}{\partial \tilde{p}}(\rho^n \tilde{L}^n_{\geq -k+1}) \varepsilon_1 \frac{\partial}{\partial \tilde{x}}(\rho \tilde{L}) - \varepsilon_1 \frac{\partial}{\partial \tilde{x}}(\rho^n \tilde{L}^n_{\geq -k+1}) \rho \frac{\partial}{\partial \tilde{p}}(\rho \tilde{L}) \right) \tag{44}$$

or

$$\rho^{k+n+1} \varepsilon_2 \varepsilon_1 \frac{\partial}{\partial \tilde{t}_n}(\tilde{L}) = \tilde{p}^k \left(\frac{\partial}{\partial \tilde{p}} \tilde{L}^n_{\geq -k+1} \frac{\partial}{\partial \tilde{x}} \tilde{L} - \frac{\partial}{\partial \tilde{x}} \tilde{L}^n_{\geq -k+1} \frac{\partial}{\partial \tilde{p}} \tilde{L} \right). \tag{45}$$

It is easy to see that if $\rho^{k+n+1} \varepsilon_2 \varepsilon_1 = 1$ then the hierarchy (39) with Lax operator (41) is invariant under the change of variables and we have a consistent nonlocal reduction.

Since $\rho^2 = 1$, we have one condition $\rho^k \varepsilon_2 \varepsilon_1 = 1$ for all odd n and another condition $\rho^{k+1} \varepsilon_2 \varepsilon_1 = 1$ for all even n. Thus, to consider nonlocal reduction of the whole hierarchy (39) we have to split it into two hierarchies

$$\frac{\partial L}{\partial t_{2n}} = \left\{ L^{2n}_{\geq -k+1}; L \right\}_k, \qquad n = 1, 2, \ldots \tag{46}$$

and

$$\frac{\partial L}{\partial t_{2n-1}} = \left\{ L^{2n-1}_{\geq -k+1}; L \right\}_k, \qquad n = 1, 2, \ldots . \tag{47}$$

Taking ρ, ε_1, and ε_2 satisfying

$$\rho^{k+1} \varepsilon_2 \varepsilon_1 = 1, \tag{48}$$

we obtain nonlocal reductions for the hierarchy (46). Note that these equations admit the following conserved quantities

$$\mathcal{Q}_{2n} = \text{tr}_k L^{2n}, \qquad n = 1, 2, \ldots, \tag{49}$$

where L is given by (41).

Taking ρ, ε_1, and ε_2 satisfying

$$\rho^k \varepsilon_2 \varepsilon_1 = 1, \tag{50}$$

we obtain nonlocal reductions for the hierarchy (47). Note that these equations admit the following conserved quantities

$$\mathcal{Q}_{2n-1} = \text{tr}_k L^{2n-1}, \qquad n = 1, 2, \ldots, \tag{51}$$

where L is given by (41).

Let us give some examples. As a first example we consider the shallow water waves system. Note that from now on in all examples in the text we let $t_n = t$ for all n.

Example 5. To have nonlocal reductions of the shallow water waves system (21), we need to work with the hierarchy (46). Taking ρ, ε_1, and ε_2 satisfying (48) we obtain possible reductions.

a. S-symmetric reduction, $\rho = -1, \varepsilon_1 = -1, \varepsilon_2 = 1$, is

$$\frac{1}{2} u_t(x, t) = (u(x, t) - u(-x, t)) u_x(x, t) - u(x, t) u_x(-x, t). \tag{52}$$

It has infinitely many symmetries, with the first symmetry, the reduction of (23), being

$$\frac{1}{4}u_t(x,t) = \left(u^3(x,t) - 9u^2(x,t)u(-x,t) + 9u(x,t)u^2(-x,t) - u^3(-x,t)\right)u_x(x,t)$$
$$-\left(3u^3(x,t) - 9u^2(x,t)u(-x,t) + 3u(x,t)u^2(-x,t)\right)u_x(-x,t). \tag{53}$$

The above equations have infinitely many conserved quantities. The first two conserved quantities are

$$\mathcal{Q}_2 = \int_I (-u^2(x,t)u(-x,t) + u(x,t)u^2(-x,t))\,dx, \tag{54}$$

$$\mathcal{Q}_4 = \int_I (-u^4(x,t)u(-x,t) + 6u^3(x,t)u^2(-x,t) - 6u^2(x,t)u^3(-x,t) + u(x,t)u^4(-x,t))\,dx. \tag{55}$$

b. T-symmetric reduction, $\rho = -1, \varepsilon_1 = 1, \varepsilon_2 = -1$, is

$$\frac{1}{2}u_t(x,t) = (u(x,t) - u(x,-t))u_x(x,t) - u(x,t)u_x(x,-t). \tag{56}$$

It has infinitely many symmetries, with the first symmetry, the reduction of (23), being

$$\frac{1}{4}u_t(x,t) = \left(u^3(x,t) - 9u^2(x,t)u(x,-t) + 9u(x,t)u^2(x,-t) - u^3(x,-t)\right)u_x(x,t)$$
$$- \left(3u^3(x,t) - 9u^2(x,t)u(x,-t) + 3u(x,t)u^2(x,-t)\right)u_x(x,-t). \tag{57}$$

The above equations have infinitely many conserved quantities. The first two conserved quantities are

$$\mathcal{Q}_2 = \int_I (-u^2(x,t)u(x,-t) + u(x,t)u^2(x,-t))\,dx, \tag{58}$$

$$\mathcal{Q}_4 = \int_I (-u^4(x,t)u(x,-t) + 6u^3(x,t)u(x,-t)^2 - 6u^2(x,t)u(x,-t)^3 + 6u(x,t)u^4(x,-t))\,dx. \tag{59}$$

c. ST-symmetric reduction, $\rho = 1, \varepsilon_1 = -1, \varepsilon_2 = -1$, is

$$\frac{1}{2}u_t(x,t) = (u(x,t) + u(-x,-t))u_x(x,t) + u(x,t)u_x(-x,-t). \tag{60}$$

It has infinitely many symmetries with the first symmetry, the reduction of (23), being

$$\frac{1}{4}u_t(x,t) = \left(u^3(x,t) + 9u^2(x,t)u(-x,-t) + 9u(x,t)u^2(-x,-t) + u^3(-x,-t)\right)u_x(x,t)$$
$$+ \left(3u^3(x,t) + 9u^2(x,t)u(-x,-t) + 3u(x,t)u^2(-x,-t)\right)u_x(-x,-t). \tag{61}$$

The above equations have infinitely many conserved quantities. The first two conserved quantities are

$$\mathcal{Q}_2 = \int_I (u^2(x,t)u(-x,-t) + u^2(x,t)u(-x,-t))\,dx, \tag{62}$$

$$\mathcal{Q}_4 = \int_I (u^4(x,t)u(-x,-t) + 6u^3(x,t)u(-x,-t)^2 + 6u^2(x,t)u^3(-x,-t) + u(x,t)u^4(-x,-t))\,dx. \tag{63}$$

We can also consider the reductions for the odd symmetries of the shallow water waves equations.

Example 6. To have nonlocal reductions of the system (22), we need to work with the hierarchy (47). Taking ρ, ε_1, and ε_2, satisfying (50) we obtain one possible reduction.

ST-symmetric reduction, $\rho = 1, \varepsilon_1 = -1, \varepsilon_2 = -1$, is

$$\frac{1}{3}u_t(x,t) = \left(u(x,t)^2 + 4u(x,t)u(-x,-t) + u^2(-x,-t)\right)u_x(x,t) + \left(2u^2(x,t) + 2u(x,t)u(-x,-t)\right)u_x(-x,-t). \tag{64}$$

The above system has infinitely many symmetries and conserved quantities. The first two conserved quantities are

$$\mathcal{Q}_3 = \int_I (u^3(x,t)u(-x,-t) + 3u^2(x,t)u^2(-x,-t) + u(x,t)u(-x,-t)^3)\,dx, \tag{65}$$

$$\mathcal{Q}_5 = \int_I (u^5(x,t)u(-x,-t) + 10u^4(x,t)u^2(-x,-t) + 20u^3(x,t)u^3(-x,-t)$$
$$+ 10u^2(x,t)u^4(-x,-t) + u(x,t)u^5(-x,-t))\,dx. \tag{66}$$

Let us also consider the reductions of the Toda system.

Example 7. To have nonlocal reductions of the Toda system (27), we need to work with the hierarchy (47). Taking ρ, ε_1, and ε_2 satisfying (50) we can obtain the following reductions: S-symmetric reduction, $\rho = -1, \varepsilon_1 = -1, \varepsilon_2 = 1$; T-symmetric reduction, $\rho = -1, \varepsilon_1 = 1, \varepsilon_2 = -1$; ST-symmetric reduction, $\rho = 1, \varepsilon_1 = -1, \varepsilon_2 = -1$. For instance, the ST-symmetric reduction of the Toda system is

$$u_t(x,t) = u(x,t)u_x(-x,-t). \tag{67}$$

It has infinitely many symmetries, with the first symmetry

$$
\frac{1}{3}u_t(x,t) = 2u(x,t)\big(u(x,t)u(-x,-t) + u^2(-x,-t)\big)u_x(x,t)
$$
$$
+ u(x,t)\big(u^2(x,t) + 4u(x,t)u(-x,-t) + u^2(x,t)\big)u_x(-x,-t), \tag{68}
$$

which is the ST-symmetric reduction of the second equation in the hierarchy (39). It also has infinitely many conserved quantities. The first two conserved quantities are

$$
\mathcal{Q}_1 = \int_I (u(x,t) + u(-x,-t))\,dx, \tag{69}
$$

$$
\mathcal{Q}_3 = \int_I (u^3(x,t) + 9u^2(x,t)u(-x,-t) + 9u(x,t)u^2(-x,-t) - u^3(-x,-t))\,dx. \tag{70}
$$

Remark 3.1. We considered the reductions for the whole hierarchy (39). We can also consider nonlocal reductions just for one equation from the hierarchy. For instance, taking $k = 0$ and $n = 2$ in (39) we get the following equations:

$$
\frac{1}{2}u_t = (u+v)u_x + uv_x, \tag{71}
$$

$$
\frac{1}{2}v_t = (u+v)v_x + vu_x. \tag{72}
$$

Let us consider nonlocal reductions of these equations only. Assume that $v(x,t) = \rho u(\varepsilon_1 x, \varepsilon_2 t) = \rho u^\varepsilon$, where $\varepsilon_1^2 = \varepsilon_2^2 = 1$, and ρ is a real constant, then the above system takes form

$$
\frac{1}{2\rho}u_t = (u^\varepsilon + \frac{1}{\rho}u)u_x + uu_x^\varepsilon, \tag{73}
$$

$$
\frac{1}{2}\varepsilon_1\varepsilon_2 u_t = (u^\varepsilon + \rho u)u_x + uu_x^\varepsilon. \tag{74}
$$

For consistency, we must have $\rho = \varepsilon_1\varepsilon_2$ which is the same condition as equality (48). Therefore working with the whole hierarchy we obtain all possible cases.

3.2. Nonlocal reductions for complex valued dependent variables

In this section we assume that all dependent variables are complex valued functions. To study reductions of such equations it is convenient to introduce a constant in the Lax equation (39) by setting new time $t' = at$, $a \in \mathbb{C}$. So we consider a hierarchy

$$
a\frac{\partial L}{\partial t_n} = \big\{L^n_{\geq -k+1}; L\big\}_k, \qquad n = 1, 2, \dots. \tag{75}
$$

Note that after the change of the time variable we shall use again t to denote new time.

The reduction is of the form

$$
v(x,t) = \rho \bar{u}(\varepsilon_1 x, \varepsilon_2 t) = \rho \bar{u}^\varepsilon. \tag{76}
$$

To find admissible reductions we use the following change of variables

$$
\tilde{x} = \varepsilon_1 x, \quad \tilde{t}_n = \varepsilon_2 t_n, \quad \tilde{p} = \rho p. \tag{77}
$$

In new variables the Lax operator becomes

$$
L = \frac{\rho}{\tilde{p}}(\rho\tilde{p} - u^\varepsilon)(\rho\tilde{p} - \rho\bar{u}) = \rho\bar{L}', \tag{78}
$$

where $L' = \frac{1}{\tilde{p}}(\tilde{p} - u)(\tilde{p} - \rho\bar{u}^\varepsilon)$ and the Lax Eq. (39) takes the form

$$
a\varepsilon_2\frac{\partial}{\partial\tilde{t}_n}(\rho\bar{L}') = (\rho\tilde{p})^k\left(\rho\frac{\partial}{\partial\tilde{p}}(\rho^n(\bar{L}')^n_{\geq -k+1})\varepsilon_1\frac{\partial}{\partial\tilde{x}}(\rho\bar{L}') - \varepsilon_1\frac{\partial}{\partial\tilde{x}}(\rho^n(\bar{L}')^n_{\geq -k+1})\rho\frac{\partial}{\partial\tilde{p}}(\rho\bar{L}')\right). \tag{79}
$$

Taking complex conjugates of both sides in the above equality we can get

$$
\bar{a}\rho^{k+n+1}\varepsilon_2\varepsilon_1\frac{\partial}{\partial\tilde{t}_n}(L') = \tilde{p}^k\left(\frac{\partial}{\partial\tilde{p}}(L')^n_{\geq -k+1}\frac{\partial}{\partial\tilde{x}}L' - \frac{\partial}{\partial\tilde{x}}(L')^n_{\geq -k+1}\frac{\partial}{\partial\tilde{p}}L'\right). \tag{80}
$$

If $\bar{a}\rho^{k+n+1}\varepsilon_2\varepsilon_1 = a$ then the hierarchy (75) is invariant under the change of variables and we have a consistent nonlocal reduction.

Since $\rho^2 = 1$ we have one condition $\bar{a}\rho^k\varepsilon_2\varepsilon_1 = a$ for all odd n and another condition $\bar{a}\rho^{k+1}\varepsilon_2\varepsilon_1 = a$ for all even n. Thus, to consider local reduction of the hierarchy (75) we have to split it into two hierarchies (46) and (47) as in the real case. Let us give one example of the reduction for complex valued dependent variables.

Example 8. We take $k = 0$ and consider the hierarchy (75) with n even. The first system corresponding to $n = 2$ is

$$
\begin{aligned}
au_t &= 2(u+v)u_x + 2uv_x, \\
av_t &= 2(u+v)v_x + 2vu_x.
\end{aligned}
\tag{81}
$$

Taking ρ, ε_1, and ε_2 satisfying $\bar{a}\rho\varepsilon_2\varepsilon_1 = a$ we obtain possible reductions.

a. S-symmetric reduction, $\varepsilon_1 = -1$, $\varepsilon_2 = 1$, is

$$
\frac{a}{2}u_t(x,t) = (u(x,t) + \rho\bar{u}(-x,t))u_x(x,t) + \rho u(x,t)\bar{u}_x(-x,t),
\tag{82}
$$

where we can take $\rho = 1$ and the constant a equals to a pure imaginary number or $\rho = -1$ and the constant a equals to a real number. The above equation has infinitely many symmetries and conserved quantities. The first two conserved quantities are

$$
\mathcal{Q}_2 = \int_I (-u^2(x,t)\rho\bar{u}(-x,t) + u(x,t)\rho\bar{u}^2(-x,t))\,dx,
\tag{83}
$$

$$
\mathcal{Q}_4 = \int_I (-u^4(x,t)\rho\bar{u}(-x,t) + 6u^3(x,t)\rho\bar{u}^2(-x,t) - 6u^2(x,t)\rho\bar{u}^3(-x,t) + u(x,t)\rho\bar{u}^4(-x,t))\,dx.
\tag{84}
$$

b. T-symmetric reduction, $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, is

$$
\frac{a}{2}u_t(x,t) = (u(x,t) + \rho\bar{u}(x,-t))u_x(x,t) + \rho u(x,t)\bar{u}_x(x,-t),
\tag{85}
$$

where we can take $\rho = 1$ and the constant a equals to a pure imaginary number or $\rho = -1$ and the constant a equals to a real number. The above equation has infinitely many symmetries and conserved quantities. The first two conserved quantities are

$$
\mathcal{Q}_2 = \int_I (-u^2(x,t)\rho\bar{u}(x,-t) + u(x,t)\rho\bar{u}^2(x,-t))\,dx,
\tag{86}
$$

$$
\mathcal{Q}_4 = \int_I (-u^4(x,t)\rho\bar{u}(x,-t) + 6u^3(x,t)\rho\bar{u}^2(x,-t) - 6u^2(x,t)\rho\bar{u}^3(x,-t) + u(x,t)\rho\bar{u}^4(x,-t))\,dx.
\tag{87}
$$

c. ST-symmetric reduction, $\varepsilon_1 = -1$, $\varepsilon_2 = -1$, is

$$
\frac{a}{2}u_t(x,t) = (u(x,t) + \rho\bar{u}(-x,-t))u_x(x,t) + \rho u(x,t)\bar{u}_x(-x,-t),
\tag{88}
$$

where we can take $\rho = 1$ and the constant a equals to a real number or $\rho = -1$ and the constant a equals to a pure imaginary number. The above equation has infinitely many symmetries and conserved quantities. The first two conserved quantities are

$$
\mathcal{Q}_2 = \int_I (-u^2(x,t)\rho\bar{u}(-x,-t) + u(x,t)\rho\bar{u}^2(-x,-t))\,dx,
\tag{89}
$$

$$
\mathcal{Q}_4 = \int_I (-u^4(x,t)\rho\bar{u}(-x,-t) + 6u^3(x,t)\rho\bar{u}^2(-x,-t) - 6u^2(x,t)\rho\bar{u}^3(-x,-t) + u(x,t)\rho\bar{u}^4(-x,-t))\,dx.
\tag{90}
$$

4. Reductions with a general Lax operator

To study nonlocal reductions of the Lax Eq. (17) with the general Lax operator (19), we can divide the variables u_i into pairs and set variables that do not have a pair to zero. For each pair of variables we consider a relation similar to (40) and proceed as before. As an example let us consider two special cases.

4.1. Three component Lax operator

In this section we take the Lax operator

$$
L = \frac{1}{p}(p-u)(p-v)(p-w),
\tag{91}
$$

and consider the hierarchy (17) with $m = 1$. Thus we have

$$
\frac{\partial L}{\partial t_n} = \left\{ L^{\frac{1}{2}+n}_{\geq -k+1}; L \right\}_k, \qquad n = 1, 2, \ldots.
\tag{92}
$$

4.1.1. Real valued dependent variables

First we study reductions for real valued dependent variables. A nonlocal reduction has a form

$$v(x, t) = \rho u(\varepsilon_1 x, \varepsilon_2 t) = \rho u^\varepsilon. \tag{93}$$

To find admissible reductions we use the following change of variables

$$\tilde{x} = \varepsilon_1 x, \quad \tilde{t}_n = \varepsilon_2 t_n, \quad \tilde{p} = \rho p. \tag{94}$$

In new variables the Lax operator becomes

$$L = \frac{1}{\rho \tilde{p}}(\rho \tilde{p} - u^\varepsilon)(\rho \tilde{p} - \rho u)(\rho \tilde{p} - w^\varepsilon) = \tilde{L}, \tag{95}$$

where $\tilde{L} = \frac{1}{\tilde{p}}(\tilde{p} - u)(\tilde{p} - \rho u^\varepsilon)(\tilde{p} - \rho w^\varepsilon)$. Also in new variables we have $L^{\frac{1}{2}} = \rho \tilde{L}^{\frac{1}{2}}$.

The Lax Eq. (92) takes the form

$$\varepsilon_2 \frac{\partial}{\partial \tilde{t}_n} \tilde{L} = (\rho \tilde{p})^k \left(\rho \frac{\partial}{\partial \tilde{p}} \rho \tilde{L}_{\geq -k+1}^{\frac{1}{2}+n} \varepsilon_1 \frac{\partial}{\partial \tilde{x}} \tilde{L} - \varepsilon_1 \frac{\partial}{\partial \tilde{x}} \rho \tilde{L}_{\geq -k+1}^{\frac{1}{2}+n} \rho \frac{\partial}{\partial \tilde{p}} \tilde{L} \right), \tag{96}$$

or

$$\rho^k \varepsilon_2 \varepsilon_1 \frac{\partial}{\partial \tilde{t}_n} \tilde{L} = \tilde{p}^k \left(\frac{\partial}{\partial \tilde{p}} \tilde{L}_{\geq -k+1}^{\frac{1}{2}+n} \frac{\partial}{\partial \tilde{x}} \tilde{L} - \frac{\partial}{\partial \tilde{x}} \tilde{L}_{\geq -k+1}^{\frac{1}{2}+n} \frac{\partial}{\partial \tilde{p}} \tilde{L} \right). \tag{97}$$

Now we set $w = 0$ in the Eq. (92) and $w^\varepsilon = 0$ in the Eq. (97). It is easy to see that under these conditions the hierarchy (92) is invariant under the change of variables if

$$\rho^k \varepsilon_2 \varepsilon_1 = 1. \tag{98}$$

So, we have a consistent nonlocal reduction if the above condition is satisfied.

Example 9. Let us consider reductions for the system (32). The condition (98) takes the form $\varepsilon_2 \varepsilon_1 = 1$. Thus we can have only *ST*-symmetric reduction. Taking $\varepsilon_1 = -1$ and $\varepsilon_2 = -1$ we get *ST*-symmetric nonlocal reduction. For $\rho = 1$, the reduced equation is

$$\frac{8}{3} u_t = \left(-u^2(x, t) + u^2(-x, -t) + 4u(x, t)u(-x, -t) \right) u_x(x, t) + \left(2u^2(x, t) + 2u(x, t)u(-x, -t) \right) u_x(-x, -t). \tag{99}$$

The above system has infinitely many generalized symmetries and conserved quantities. The first two conserved quantities are

$$\mathcal{Q}_1 = \int_I (u^2(x, t) + u^2(-x, -t) - 2u(x, t)u(-x, -t)) \, dx, \tag{100}$$

$$\mathcal{Q}_2 = \int_I (u^4(x, t) + u^4(-x, -t) - 4u^3(x, t)u(-x, -t) - 4u(x, t)u^3(-x, -t) + 6u^2(x, t)u^2(-x, -t)) \, dx. \tag{101}$$

4.1.2. Complex valued dependent variables

In this section we assume that all variables in the Lax Eq. (92) are complex valued functions. As in the case $N = 2$, we transform the Lax equation to the form

$$a \frac{\partial L}{\partial t_n} = \left\{ L_{\geq -k+1}^{\frac{1}{2}+n}; L \right\}_k, \qquad n = 1, 2, \ldots. \tag{102}$$

The reduction is of the form

$$v(x, t) = \rho \bar{u}(\varepsilon_1 x, \varepsilon_2 t) = \rho \bar{u}^\varepsilon. \tag{103}$$

To find admissible reductions we use the following change of variables

$$\tilde{x} = \varepsilon_1 x, \quad \tilde{t}_n = \varepsilon_2 t_n, \quad \tilde{p} = \rho p, \tag{104}$$

for the Lax Eq. (102).

In new variables the Lax operator becomes

$$L = \frac{1}{\rho \tilde{p}}(\rho \tilde{p} - \bar{u}^\varepsilon)(\rho \tilde{p} - \rho \bar{u})(\rho \tilde{p} - w^\varepsilon) = \bar{L}', \tag{105}$$

where $L' = \frac{1}{\tilde{p}}(\tilde{p} - u)(\tilde{p} - \rho \bar{u}^\varepsilon)(\tilde{p} - \rho \bar{w}^\varepsilon)$. Also in new variables we have $L^{\frac{1}{2}} = \rho \bar{L}'^{\frac{1}{2}}$.

The Lax Eq. (102) takes the form

$$a\varepsilon_2 \frac{\partial}{\partial \tilde{t}_n} \bar{L}' = (\rho \tilde{p})^k \left(\rho \frac{\partial}{\partial \tilde{p}} \rho \bar{L}'_{\geq -k+1}^{\frac{1}{2}+n} \varepsilon_1 \frac{\partial}{\partial \tilde{x}} \bar{L}' - \varepsilon_1 \frac{\partial}{\partial \tilde{x}} \rho \bar{L}'_{\geq -k+1}^{\frac{1}{2}+n} \rho \frac{\partial}{\partial \tilde{p}} \bar{L}' \right). \tag{106}$$

Taking complex conjugates of both sides in the above equality we get

$$\bar{a}\rho^{k+1}\varepsilon_2\varepsilon_1\frac{\partial}{\partial\tilde{t}_n}L' = \tilde{p}^k\left(\frac{\partial}{\partial\tilde{p}}L'^{\frac{1}{2}+n}_{\geq-k+1}\frac{\partial}{\partial\tilde{x}}L' - \frac{\partial}{\partial\tilde{x}}L'^{\frac{1}{2}+n}_{\geq-k+1}\frac{\partial}{\partial\tilde{p}}L'\right). \tag{107}$$

Now we set $w = 0$ in the Eq. (92) and $w^\varepsilon = 0$ in the Eq. (97). It is easy to see that under these conditions the hierarchy (92) is invariant under the change of variables if

$$\bar{a}\rho^k\varepsilon_2\varepsilon_1 = a. \tag{108}$$

So, we have a consistent nonlocal reduction if the above condition is satisfied.

Example 10. Let us consider reductions for the modified system (32) (the time is redefined to introduce the multiple a). The condition (108) takes the form $\bar{a}\varepsilon_2\varepsilon_1 = a$. Thus we can have S-symmetric, T-symmetric, and ST-symmetric reductions. For instance, taking a purely imaginary, $\varepsilon_1 = -1$, and $\varepsilon_2 = 1$ we get S-symmetric nonlocal reduction. For $\rho = 1$, the reduced equation is

$$\frac{8}{3}au_t = \left(-u^2(x,t) + \bar{u}^2(-x,t) + 4u(x,t)\bar{u}(-x,t)\right)u_x(x,t) + \left(2u^2(x,t) + 2u(x,t)\bar{u}(-x,t)\right)\bar{u}_x(-x,t). \tag{109}$$

The above system has infinitely many generalized symmetries and conserved quantities. The first two conserved quantities are

$$\mathcal{Q}_1 = \int_I (u^2(x,t) + \bar{u}^2(-x,t) - 2u(x,t)\bar{u}(-x,t))\,dx, \tag{110}$$

$$\mathcal{Q}_2 = \int_I (u^4(x,t) + u^4(-x,t) - 4u^3(x,t)\bar{u}(-x,t) - 4u(x,t)\bar{u}^3(-x,t) + 6u^2(x,t)\bar{u}^2(-x,t))\,dx. \tag{111}$$

4.2. Four component Lax operator

In this section we take the Lax operator

$$L = \frac{1}{p}(p-u)(p-u_1)(p-v)(p-v_1), \tag{112}$$

and consider the hierarchy (17) with $m = 0$, for simplicity. Thus we have

$$\frac{\partial L}{\partial t_n} = \left\{L^n_{\geq-k+1}; L\right\}_k, \qquad n = 1, 2, \dots. \tag{113}$$

4.2.1. Real valued dependent variables

First, we study reductions for real valued dependent variables. A nonlocal reduction has a form

$$u_1(x,t) = \rho u(\varepsilon_1 x, \varepsilon_2 t) = \rho u^\varepsilon, \quad v_1(x,t) = \rho v(\varepsilon_1 x, \varepsilon_2 t) = \rho v^\varepsilon. \tag{114}$$

To find admissible reductions we use the following change of variables

$$\tilde{x} = \varepsilon_1 x, \quad \tilde{t}_n = \varepsilon_2 t_n, \quad \tilde{p} = \rho p. \tag{115}$$

In new variables the Lax operator becomes

$$L = \frac{1}{\rho\tilde{p}}(\rho\tilde{p} - u^\varepsilon)(\rho\tilde{p} - \rho u)(\rho\tilde{p} - v^\varepsilon)(\rho\tilde{p} - \rho v) = \rho\tilde{L}, \tag{116}$$

where $\tilde{L} = \frac{1}{\tilde{p}}(\tilde{p} - u)(\tilde{p} - \rho u^\varepsilon)(\rho\tilde{p} - v^\varepsilon)(\rho\tilde{p} - \rho v)$. The Lax Eq. (113) takes the form

$$\varepsilon_2\frac{\partial}{\partial\tilde{t}_n}(\rho\tilde{L}) = (\rho\tilde{p})^k\left(\rho\frac{\partial}{\partial\tilde{p}}(\rho^n\tilde{L}^n)_{\geq-k+1}\varepsilon_1\frac{\partial}{\partial\tilde{x}}(\tilde{L}) - \varepsilon_1\frac{\partial}{\partial\tilde{x}}(\rho^n\tilde{L}^n)_{\geq-k+1}\rho\frac{\partial}{\partial\tilde{p}}(\tilde{L})\right) \tag{117}$$

or

$$\rho^{n+k+1}\varepsilon_2\varepsilon_1\frac{\partial}{\partial\tilde{t}_n}(\tilde{L}) = \tilde{p}^k\left(\frac{\partial}{\partial\tilde{p}}\tilde{L}^n_{\geq-k+1}\frac{\partial}{\partial\tilde{x}}\tilde{L} - \frac{\partial}{\partial\tilde{x}}\tilde{L}^n_{\geq-k+1}\frac{\partial}{\partial\tilde{p}}\tilde{L}\right). \tag{118}$$

If

$$\rho^{n+k+1}\varepsilon_2\varepsilon_1 = 1 \tag{119}$$

then the hierarchy (113) is invariant under change of variables and we have a consistent nonlocal reduction. As in the case $N = 2$, to have integrable hierarchy of nonlocal systems we have to split our hierarchy into two hierarchies: one with even n and another with odd n.

Example 11. Let us consider reductions for the system (36). It follows that for nonlocal reductions we have to consider the systems of hierarchy (112), corresponding to odd n. The condition (119) takes the form $\varepsilon_2 \varepsilon_1 = 1$. Thus we can have only *ST*-symmetric reduction, $\varepsilon_1 = -1, \varepsilon_2 = -1$. Taking $\rho = 1$ and setting $u_1(x,t) = u(-x,-t) = u^\varepsilon$ and $v_1(x,t) = v(-x,-t) = v^\varepsilon$ we obtain ST-symmetric nonlocal reduction.

$$
\begin{aligned}
u_t &= (u^\varepsilon v + u^\varepsilon v^\varepsilon + v v^\varepsilon)u_x + (uv + uv^\varepsilon)u_x^\varepsilon + (uu^\varepsilon + uv^\varepsilon)v_x + (uu^\varepsilon + uv)v_x^\varepsilon, \\
v_t &= (uu^\varepsilon + uv^\varepsilon + u^\varepsilon v^\varepsilon)v_x + (u^\varepsilon v + v v^\varepsilon)u_x + (uv + v v^\varepsilon)u_x^\varepsilon + (uv + u^\varepsilon v)v_x^\varepsilon.
\end{aligned}
\tag{120}
$$

The above system has infinitely many generalized symmetries and conserved quantities. The first two conserved quantities are

$$
\mathcal{Q}_1 = \int_I u u^\varepsilon v v^\varepsilon \, dx,
\tag{121}
$$

$$
\mathcal{Q}_2 = \int_I \left(u^2(u^\varepsilon)^2 v^2 v^\varepsilon + u^2(u^\varepsilon)^2 v(v^\varepsilon)^2 + u^2 u^\varepsilon v^2(v^\varepsilon)^2 + u(u^\varepsilon)^2 v^2(v^\varepsilon)^2 \right) dx.
\tag{122}
$$

4.2.2. Complex valued dependent variables

In this section we assume that all variables in the Lax Eq. (113) are complex valued functions. As in the case $N = 2$, we transform the Lax equation to the form

$$
a\frac{\partial L}{\partial t_n} = \left\{ L^n_{\geq -k+1}; L \right\}_k, \qquad n = 1, 2, \ldots .
\tag{123}
$$

The reduction is of the form

$$
u_1(x,t) = \rho\bar{u}(\varepsilon_1 x, \varepsilon_2 t) = \rho\bar{u}^\varepsilon, \qquad v_1(x,t) = \rho\bar{v}(\varepsilon_1 x, \varepsilon_2 t) = \rho\bar{v}^\varepsilon.
\tag{124}
$$

To find admissible reductions we use the following change of variables

$$
\tilde{x} = \varepsilon_1 x, \quad \tilde{t}_n = \varepsilon_2 t_n, \quad \tilde{p} = \rho p,
\tag{125}
$$

for the Lax Eq. (123).

In new variables the Lax operator becomes

$$
L = \frac{\rho}{\tilde{p}}(\rho\tilde{p} - u^\varepsilon)(\rho\tilde{p} - \rho\bar{u})(\rho\tilde{p} - v^\varepsilon)(\rho\tilde{p} - \rho\bar{v}) = \rho\bar{L}',
\tag{126}
$$

where $L' = \frac{1}{\tilde{p}}(\tilde{p} - u)(\tilde{p} - \rho\bar{u}^\varepsilon)(\tilde{p} - v)(\tilde{p} - \rho\bar{v}^\varepsilon)$. The Lax Eq. (123) takes the form

$$
a\varepsilon_2 \frac{\partial}{\partial \tilde{t}_n}(\rho\bar{L}') = (\rho\tilde{p})^k \left(\rho\frac{\partial}{\partial \tilde{p}}(\rho^n(\bar{L}')^n)_{\geq -k+1}\varepsilon_1 \frac{\partial}{\partial \tilde{x}}(\rho\bar{L}') - \varepsilon_1 \frac{\partial}{\partial \tilde{x}}(\rho^n(\bar{L}')^n)_{\geq -k+1}\rho\frac{\partial}{\partial \tilde{p}}(\rho\bar{L}') \right).
\tag{127}
$$

Taking complex conjugates of both sides in the above equality we get

$$
\bar{a}\rho^{k+n+1}\varepsilon_2\varepsilon_1 \frac{\partial}{\partial \tilde{t}_n}(L') = \tilde{p}^k \left(\frac{\partial}{\partial \tilde{p}}(L')^n_{\geq -k+1}\frac{\partial}{\partial \tilde{x}}L' - \frac{\partial}{\partial \tilde{x}}(L')^n_{\geq -k+1}\frac{\partial}{\partial \tilde{p}}L' \right).
\tag{128}
$$

If

$$
\bar{a}\rho^{k+n+1}\varepsilon_2\varepsilon_1 = a,
\tag{129}
$$

then the hierarchy (123) is invariant under the change of variables and we have a consistent nonlocal reduction.

Since $\rho^2 = 1$ we have one condition $\bar{a}\rho^k\varepsilon_2\varepsilon_1 = a$ for all odd n and another condition $\bar{a}\rho^{k+1}\varepsilon_2\varepsilon_1 = a$ for all even n. Thus, to consider nonlocal reduction of the hierarchy (75) we have to split it into two hierarchies as in the real case.

Example 12. Let us consider reductions for the modified system (36) (the time is redefined to introduce the multiple a). It follows that for nonlocal reductions we have to consider the systems of hierarchy (123) corresponding to odd n. The condition (129) takes the form $\bar{a}\varepsilon_2\varepsilon_1 = a$. Thus we can have *S*-symmetric, *T*-symmetric, and *ST*-symmetric reductions. For instance, taking a purely imaginary and $\varepsilon_1 = -1, \varepsilon_2 = 1$ we get *S*-symmetric nonlocal reduction. Taking $\rho = 1$ and setting $u_1(x,t) = \bar{u}(-x,t) = \bar{u}^\varepsilon$ and $v_1(x,t) = \bar{v}(-x,t) = \bar{v}^\varepsilon$ we can write the reduced equation as

$$
\begin{aligned}
au_t &= \left(\bar{u}^\varepsilon v + \bar{u}^\varepsilon\bar{v}^\varepsilon + v\bar{v}^\varepsilon\right)u_x + \left(uv + u\bar{v}^\varepsilon\right)\bar{u}_x^\varepsilon + \left(u\bar{u}^\varepsilon + u\bar{v}^\varepsilon\right)v_x + \left(u\bar{u}^\varepsilon + uv\right)\bar{v}_x^\varepsilon, \\
av_t &= \left(u\bar{u}^\varepsilon + u\bar{v}^\varepsilon + \bar{u}^\varepsilon\bar{v}^\varepsilon\right)v_x + \left(\bar{u}^\varepsilon v + v\bar{v}^\varepsilon\right)u_x + \left(uv + v\bar{v}^\varepsilon\right)\bar{u}_x^\varepsilon + \left(uv + \bar{u}^\varepsilon v\right)\bar{v}_x^\varepsilon.
\end{aligned}
\tag{130}
$$

The above system has infinitely many generalized symmetries and conserved quantities. The first two conserved quantities are

$$
\mathcal{Q}_1 = \int_I u\bar{u}^\varepsilon v\bar{v}^\varepsilon \, dx,
\tag{131}
$$

$$
\mathcal{Q}_2 = \int_I \left(u^2(\bar{u}^\varepsilon)^2 v^2\bar{v}^\varepsilon + u^2(\bar{u}^\varepsilon)^2 v(\bar{v}^\varepsilon)^2 + u^2\bar{u}^\varepsilon v^2(\bar{v}^\varepsilon)^2 + u(\bar{u}^\varepsilon)^2 v^2(\bar{v}^\varepsilon)^2 \right) dx.
\tag{132}
$$

5. Conclusion

We have found all possible nonlocal reductions of the hydrodynamic type of equations in $(1 + 1)$ dimensions. Such reductions are possible only for all even numbered systems. We have given examples of nonlocal reductions for $N = 2$ and $N = 4$ when the dynamical variables are real and complex valued functions. For the complex valued dynamical variables we have all kinds of nonlocal reductions, namely space reflection, time reflection, and space-time reflection. The new nonlocal systems admit Lax representations and possess infinitely many conserved quantities. We gave some examples of these conserved quantities of the new nonlocal reduced systems.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Metin Gürses: Conceptualization, Methodology, Supervision, Investigation, Visualization, Writing - review & editing. **Aslı Pekcan:** Methodology, Validation, Formal analysis, Writing - original draft, Writing - review & editing. **Kostyantyn Zheltukhin:** Methodology, Validation, Formal analysis, Writing - original draft, Writing - review & editing.

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