



GENERATING FUNCTION OF LAGUERRE POLYNOMIALS

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Rodriguez formula for Laguerre polynomials is defined as

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t})$$

Evaluating the first three derivatives of $ws^n = e^{-t}t^n$ we get the following results:

$$\frac{d}{dt}(t^n e^{-t}) = nt^{n-1}e^{-t} - t^n e^{-t}$$

$$\frac{d^2}{dt^2}(t^n e^{-t}) = n(n-1)t^{n-2}e^{-t} - 2nt^{n-1}e^{-t} + t^n e^{-t}$$

$$\frac{d^3}{dt^3}(t^n e^{-t}) = n(n-1)(n-2)t^{n-3}e^{-t} - 3n(n-1)t^{n-2}e^{-t} + 3nt^{n-1}e^{-t} - t^n e^{-t}$$

Then

$$\frac{d^n}{dt^n}(t^n e^{-t}) = \sum_{k=0}^n C(n, k)(-1)^k t^k \frac{n!}{k!} e^{-t}.$$

Then we find

$$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) = \sum_{k=0}^n C(n, k)(-1)^k \frac{t^k}{k!}.$$

Hence we evaluate the generating function as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} L_n(t)w^n &= \sum_{n=0}^{\infty} \sum_{k=0}^n C(n, k)(-1)^k \frac{t^k}{k!} w^n \\ &= (-1)^0 C(0, 0) \frac{t^0}{0!} w^0 + \sum_{k=0}^1 C(1, k)(-1)^k \frac{t^k}{k!} w + \dots + \sum_{k=0}^n C(n, k)(-1)^k \frac{t^k}{k!} w^n + \dots \\ &= (-1)^0 \frac{t^0}{0!} [C(0, 0)w^0 + C(1, 0)w + \dots + C(n, 0)w^n + \dots] \\ &+ (-1)^1 \frac{t}{1!} [C(1, 1)w + C(2, 1)w^2 + \dots + C(n, 1)w^n + \dots] \\ &+ (-1)^2 \frac{t^2}{2!} [C(2, 2)w^2 + C(3, 2)w^3 + \dots + C(n, 2)w^n + \dots] \\ &+ \dots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \sum_{n=k}^{\infty} C(n, k)w^n. \end{aligned}$$

Let us observe the following identity

$$\begin{aligned}
\sum_{n=k}^{\infty} C(n, k)w^n &= \sum_{n=0}^{\infty} C(n+k, k)w^{n+k} \\
&= w^k \sum_{n=0}^{\infty} C(n+k, k)w^n \\
&= w^k \sum_{n=0}^{\infty} C(n+k, n)w^n \\
&= w^k [C(k, 0)w^0 + C(k+1, 1)w + \dots + C(k+n, n)w^n + \dots].
\end{aligned}$$

Consider the function $f(w) = \frac{1}{(1-w)^{k+1}} = (1-w)^{-k-1}$. The first three derivatives of f at 0 are $\frac{df}{dw}(0) = k+1$, $\frac{d^2f}{dw^2}(0) = (k+1)(k+2)$, $\frac{d^3f}{dw^3}(0) = (k+1)(k+2)(k+3)$. Then the Maclaurien series of f is

$$\begin{aligned}
f(w) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} w^n \\
&= 1 + (k+1)w + \frac{(k+1)(k+2)}{2!} w^2 + \dots + \frac{\prod_{m=1}^n (k+m)}{n!} w^n + \dots \\
&= \sum_{n=0}^{\infty} C(n+k, n)w^n.
\end{aligned}$$

Then we conclude that the generating function of Laguerre polynomials for $\nu = 0$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n(t)w^n &= \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \frac{w^k}{(1-w)^{k+1}} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{tw}{1-w}\right)^k}{k!} \frac{1}{1-w} \\
&= e^{-\frac{tw}{1-w}} \frac{1}{1-w}.
\end{aligned}$$

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GENERATING FUNCTION FOR HERMITE POLYNOMIALS

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GENERATING FUNCTION FOR HERMITE POLYNOMIALS

We know that the rodriguez type formula for hermite polynomials is $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ by taking $s(x) = 1$ and $w(x) = e^{-x^2}$ ($x \in [-\infty, \infty]$). This can be found in the book [1].

We define a function f as e^{-x^2} . This is just for convenience. $\frac{d^n}{dx^n} e^{-x^2} = f^{(n)}(x) = (-1)^n e^{-x^2} H_n(x)$. We know f is entire function so we can use the taylor series expansion of f at 0.

We write the taylor expansion of $f(x+t)$ for any t .

$$f(x+t) = \sum_0^{\infty} \frac{t^n}{n!} f^{(n)}(x) = \sum_0^{\infty} \frac{t^n}{n!} (-1)^n e^{-x^2} H_n(x)$$

By taking $-t$ instead of t , we can get the following equation;

$$f(x-t) = \sum_0^{\infty} \frac{t^n}{n!} e^{-x^2} H_n(x)$$

Multiplying both sides by e^{x^2} , we get the following;

$$f(x-t)e^{x^2} = \sum_0^{\infty} \frac{t^n}{n!} H_n(x)$$

Remember that $f(x) = e^{-x^2}$. Thus, we have

$$e^{-(x-t)^2} e^{x^2} = e^{2xt-t^2} = \sum_0^{\infty} \frac{t^n}{n!} H_n(x)$$

In last equation, $H_n(x)$ correponds $\frac{\delta^n}{\delta t^n} e^{2xt-t^2}$ at $t = 0$ (From Taylor series expansion). Hence, we can get all $H_n(x)$ from this function e^{2xt-t^2} .

As a result, e^{2xt-t^2} is the generating function for hermite polynomials $H_n(x)$.

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MATH 543 BONUS PROJECT 1

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ABSTRACT. In this paper, we will obtain a generating function for Tchebychev Polynomials of the second kind, denoted $U_n(x)$, by the recursion formula they satisfy. The recursion formula will also be obtained.

1. INTRODUCTION

We know from the lecture that if $\omega(x)=\sqrt{1-x^2}$ and $s(x)=1-x^2$ then the Rodriguez formula is

$$(1.1) \quad \frac{1}{w(x)} \frac{d^n}{dx^n} (\omega(x)s^n(x)) = U_n(x) \text{ for } n = 0, 1, 2, \dots$$

and

$$(1.2) \quad \|U_n(x)\|^2 = \int_{-1}^1 \omega(x)U_n^2(x)dx = \frac{\pi}{2} \text{ hence } \|U_n(x)\| = \sqrt{\frac{\pi}{2}}$$

Computing some of the polynomials by the Rodriguez formula we get

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \\ U_4(x) &= 16x^4 - 12x^2 + 1 \\ U_5(x) &= 32x^5 - 32x^3 + 6x \end{aligned}$$

Thus, one can easily show that (by induction) the leading coefficient of $U_n(x)$ is 2^n and the coefficient of x^{n-1} is 0. That is, if $U_n(x) = k_n x^n + k'_n x^{n-1} + \dots$ then $k_n = 2^n$ and $k'_n = 0$.

2. FINDING GENERATING FUNCTION

Let $h_n = \|U_n(x)\|^2 = \int_{-1}^1 \omega(x)U_n^2(x)dx$ then $h_n = \frac{\pi}{2}$ for $n = 0, 1, 2, \dots$ also let k_n and k'_n is the same as above. Then we know from the lecture that $U_n(x)$ satisfy the following recursion relation:

$$(2.1) \quad U_{n+1}(x) - \frac{k_{n+1}}{k_n} x U_n(x) = a_n U_n(x) + a_{n-1} U_{n-1}(x)$$

where

$$a_{n-1}^n = -\frac{k_{n-1}k_{n+1}}{k_n^2} \frac{h_n}{h_{n-1}} \text{ and } a_n^n = \frac{k'_{n+1}}{k_n} - \frac{k_{n+1}}{k_n^2} k'_n$$

Thus we have

$$a_{n-1}^n = -\frac{2^{n-1}2^{n+1}}{2^{2n}} \frac{\frac{\pi}{2}}{\frac{\pi}{2}} = -1 \text{ and } a_n^n = \frac{0}{2^n} - \frac{2^{n+1}}{2^{2n}} 0 = 0$$

Hence the recursion relation (2.1) turns into this recursion formula

$$(2.2) \quad U_{n+1}(x) - 2xU_n(x) = -U_{n-1}(x)$$

Multiplying the recursion formula (2.2) by t^{n+1} and summing over $n = 1$ to ∞ we get

$$(2.3) \quad \sum_{n=1}^{\infty} t^{n+1}U_{n+1}(x) - 2xt \sum_{n=1}^{\infty} t^n U_n(x) = -t^2 \sum_{n=1}^{\infty} t^{n-1}U_{n-1}(x)$$

By shifting the indices n to $n + 1$ in (2.3) we get

$$(2.4) \quad \sum_{n=0}^{\infty} t^{n+2}U_{n+2}(x) - 2xt \sum_{n=0}^{\infty} t^{n+1}U_{n+1}(x) = -t^2 \sum_{n=0}^{\infty} t^n U_n(x)$$

Since the generating function is $g(x, t) = \sum_{n=0}^{\infty} t^n U_n(x)$ note that

$g(x, t) = U_0(x) + \sum_{n=1}^{\infty} t^n U_n(x) = 1 + \sum_{n=0}^{\infty} t^{n+1}U_{n+1}(x)$ by shifting the indices and using $U_0(x) = 1$. Also,

$g(x, t) = U_0(x) + tU_1(x) + \sum_{n=2}^{\infty} t^n U_n(x) = 1 + 2xt + \sum_{n=0}^{\infty} t^{n+2}U_{n+2}(x)$ by shifting the indices and using $U_1(x) = 2x$.

So we find $\sum_{n=0}^{\infty} t^{n+1}U_{n+1}(x) = g(x, t) - 1$ and $\sum_{n=0}^{\infty} t^{n+2}U_{n+2}(x) = g(x, t) - 2xt - 1$

putting these into (2.4) we get

$$(2.5) \quad g(x, t) - 1 - 2xt - 2xt(g(x, t) - 1) = -t^2 g(x, t)$$

solving this equation for $g(x, t)$ we get

$$g(x, t) - 1 - 2xt - 2xtg(x, t) + 2xt = -t^2 g(x, t)$$

$$g(x, t) - 1 - 2xtg(x, t) = -t^2 g(x, t)$$

$$g(x, t) - 2xtg(x, t) + t^2g(x, t) = 1$$

$$g(x, t)(1 - 2xt + t^2) = 1$$

$$(2.6) \quad g(x, t) = \frac{1}{1 - 2xt + t^2}$$

Consequently, we find the generating function for Tchebychev Polynomials of the second kind, $U_n(x)$, which is shown as in (2.6)

$$g(x, t) = \sum_{n=0}^{\infty} t^n U_n(x) = \frac{1}{1 - 2xt + t^2}$$

GENERATING FUNCTION OF FIRST KIND CHEBYSHEV POLYNOMIALS

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In this project we will find generating function of first kind Chebyshev polynomials by using the identity $T_n(x) = \cos n\theta$, where $x = \cos \theta$, $\theta \in [0, \pi]$ and $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. So let us first prove this identity by induction. The Rodrigues formula of first kind Chebyshev polynomials is as follows:

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} [(1-x^2)^{n-\frac{1}{2}}]$$

First let us show the case $n = 0$.

$$T_0(x) = (1-x^2)^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} = 1 = \cos 0$$

Assume $T_n(x) = \cos n\theta$ for some fixed $n \in \mathbb{N}$. Note that $\frac{dx}{d\theta} = -\sin \theta$. Let us show for $n + 1$.

$$\begin{aligned} T_{n+1}(x) &= \frac{(-1)^{n+1} 2^{n+1} (n+1)!}{(2n+2)!} (1-x^2)^{\frac{1}{2}} \frac{d^{n+1}}{dx^{n+1}} [(1-x^2)^{n+\frac{1}{2}}] \\ &= \frac{(-1)^{n+1} 2^n (n)!}{(2n+1)!} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} \left[\left(n + \frac{1}{2}\right) (1-x^2)^{n-\frac{1}{2}} (-2x) \right] \\ &= \frac{(-1)^n 2^n (n)!}{(2n)!} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} [x(1-x^2)^{n-\frac{1}{2}}] \end{aligned}$$

Here we should observe the following identity.

$$\begin{aligned} \frac{d^n}{dx^n} [x(1-x^2)^{n-\frac{1}{2}}] &= \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}] + \frac{d^{n-1}}{dx^{n-1}} x \frac{d}{dx} [(1-x^2)^{n-\frac{1}{2}}] \\ &= 2 \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}] + \frac{d^{n-2}}{dx^{n-2}} x \frac{d^2}{dx^2} [(1-x^2)^{n-\frac{1}{2}}] \\ &= 3 \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}] + \frac{d^{n-3}}{dx^{n-3}} x \frac{d^3}{dx^3} [(1-x^2)^{n-\frac{1}{2}}] \\ &= \dots \\ &= n \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}] + x \frac{d^n}{dx^n} [(1-x^2)^{n-\frac{1}{2}}] \end{aligned}$$

Hence

$$\begin{aligned}
T_{n+1}(x) &= \frac{(-1)^n 2^n (n)!}{(2n)!} (1-x^2)^{\frac{1}{2}} \left\{ n \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}] + x \frac{d^n}{dx^n} [(1-x^2)^{n-\frac{1}{2}}] \right\} \\
&= \sin \theta \frac{(-1)^n 2^n (n)!}{(2n)!} n \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}] + \cos \theta \cos n\theta,
\end{aligned}$$

by our assumption and $x = \cos \theta$. Let us also observe the following identity:

$$\begin{aligned}
\int \cos n\theta d\theta &= \int \frac{(-1)^n 2^n n!}{(2n)!} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} [(1-x^2)^{n-\frac{1}{2}}] d\theta \\
&= \frac{(-1)^n 2^n n!}{(2n)!} \int \sin \theta \frac{d}{- \sin \theta d\theta} \left[\frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}] \right] d\theta \\
&= - \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}]
\end{aligned}$$

Therefore we get the following identity:

$$- \sin n\theta = \frac{(-1)^n 2^n n!}{(2n)!} n \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}]$$

This identity gives us the result.

$$\begin{aligned}
T_{n+1}(x) &= \sin \theta \frac{(-1)^n 2^n (n)!}{(2n)!} n \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n-\frac{1}{2}}] + \cos \theta \cos n\theta \\
&= \cos \theta \cos n\theta - \sin \theta \sin n\theta \\
&= \cos(n+1)\theta
\end{aligned}$$

This completes proof, so $T_n(x) = \cos n\theta$, where $x = \cos \theta$, $\theta \in [0, \pi]$ and $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. Hence for any n we have

$$\operatorname{Re}(t^n e^{in\theta}) = \operatorname{Re}(t^n \cos n\theta + it^n \sin n\theta) = t^n T_n(x).$$

It is natural to ask is this equality also valid for infinite sum, i.e. is the following equality valid:

$$\operatorname{Re}\left(\sum_{n=0}^{\infty} t^n e^{in\theta}\right) = \sum_{n=0}^{\infty} t^n T_n(x)$$

where $x = \cos \theta$.

Let us find the value of the infinite sum on the left hand side.

$$\sum_{n=0}^{\infty} t^n e^{in\theta} = \sum_{n=0}^{\infty} (te^{i\theta})^n = \lim_n \frac{1 - (te^{i\theta})^n}{1 - te^{i\theta}}$$

where n goes to plus infinity, but this limit is finite only if $-1 < t < 1$. Hence take $-1 < t < 1$. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n e^{in\theta} &= \frac{1}{1 - te^{i\theta}} \\
&= \frac{1}{1 - t \cos \theta - it \sin \theta} \\
&= \frac{1 - t \cos \theta + it \sin \theta}{(1 - t \cos \theta - it \sin \theta)(1 - t \cos \theta + it \sin \theta)} \\
&= \frac{1 - t \cos \theta + it \sin \theta}{1 - 2t \cos \theta + t^2}
\end{aligned}$$

where $x = \cos \theta$. Hence for $-1 < t < 1$ we get the following result and find generating function of Chebyshev polynomials of first kind.

$$\sum_{n=0}^{\infty} t^n T_n(x) = \operatorname{Re} \left(\sum_{n=0}^{\infty} (te^{i\theta})^n \right) = \frac{1 - tx}{1 - 2tx + t^2} = g(x, t).$$

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