ON A PARTICULAR TYPE OF PRODUCT MANIFOLDS AND SHEAR-FREE COSMOLOGICAL MODELS

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Abstract. Shear-free flows or observer fields are important objects of study in general relativity; stationary or rigid observers are important examples of shear-free reference frames. In this paper, we introduce a geometric structure based on a local coordinate expression of metrics admitting a shear-free reference frame. Furthermore, we investigate a large sub-class of these models ("tilted" warped products) that includes the Robertson–Walker spacetime, the Gödel spacetime and other models of Gödel type. We present a novel example of a rotating and expanding cosmological model that is contained in this class. Finally, we describe the geodesic barotropic perfect fluid solutions.

1. Introduction

In general relativity, many known exact solutions of Einstein's equations admit symmetries such as being (conformally) stationary or static. The metric of spacetimes admitting such symmetries may locally be written in certain well-known standard forms, for example, a warped product in the static case. A considerable amount of work has been invested in the question when such symmetries yield a global structure of this kind, a problem which is usually tied to issues of causality (cf. for example [19]). In the following, we will derive and analyze a standard form for spacetimes admitting a shear-free reference frame, and apply the results in the construction of cosmological models.

Notation and Conventions. Let M be a d-dimensional smooth manifold, and g a Riemannian or Lorentzian metric on M; the Levi–Civita derivative associated with g will be denoted by ∇. We will also write $v \wedge w = v \otimes w - w \otimes v$ and $v \vee w = v \otimes w + w \otimes v$ for one-forms v and w on M.
Now let V be a reference frame, i.e., a cospacelike vector field V on M with $|g(V,V)| = 1$. We will write $u := g(V, \cdot)$ and $P := g - \varepsilon u \otimes u$, where $\varepsilon = +1$ in the Riemannian and $\varepsilon = -1$ in the Lorentzian case.

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The $1+(d-1)$-decomposition of the tensor field $(\nabla u)(X,Y) := g(\nabla_X V, Y)$ (with vector fields X, Y) is then given by [18, p. 83]:

$$\nabla u = \sigma - \omega + \frac{\Theta}{d-1} P + \varepsilon u \otimes \dot{u},$$

with the *kinematical quantities*:

$$\begin{aligned} \Theta &= \operatorname{div} V && \textit{expansion scalar}, \\ \dot{u} &= g(\nabla_V V, \,\cdot\,) && \textit{acceleration}, \\ \omega &= -\frac{1}{2}\left(\mathrm{d}u + \varepsilon \dot{u} \wedge u\right) && \textit{vorticity tensor}, \\ \sigma &= \operatorname{sym}(\nabla u) - \frac{1}{2}\varepsilon \dot{u} \vee u - \frac{\Theta}{d-1} P && \textit{shear tensor}. \end{aligned}$$

Remark 1.1. Employing Einstein's summation convention, in local coordinates,

$$\Theta = \nabla_k u^k, \ \dot{u}_i = u^k \nabla_k u_i, \ \omega_{ij} = -P^k_{\ i} P^l_{\ j} \nabla_{[k} u_{l]},$$
$$\sigma_{ij} = P^k_{\ i} P^l_{\ j} \nabla_{(k} u_{l)} - \frac{\Theta}{d-1} P_{ij}.$$

Here, parantheses (brackets) denote (anti-)symmetrization. In this paper, we will use both index notation and coordinate-free notation.

We will mostly be concerned with the Lorentzian case and applications to general relativity. However, we we will also treat the case $\varepsilon = +1$ in the proof of Theorem 3.6.

2. Shear-free Reference Frames

Shear-free reference frames frequently appear in general relativity. For example, any (conformally) stationary, Hubble-isotropic [17] or rigid reference frame is shear-free. Furthermore, an important and still unsettled conjecture states that shear-free barotropic perfect fluids have vanishing vorticity or expansion [26]. Finally, it should be noted that a reference frame is shear-free iff it is spatially conformally stationary [11, Def. 3.1].
Locally, any metric admitting a shear-free reference frame can be brought into a standard form, see also [4, 14]:

Proposition 2.1. *Let (M, g) be a Riemannian or Lorentzian manifold admitting a shear-free reference frame V, $u = g(V, \cdot)$. Then about each point $p \in M$ there exists a chart (U, η) with coordinates $\eta = (x^0, x^1, \dots, x^{d-1}) = (t, x)$ such that $V = \partial_t$ and*

$$g_{ij}(t,x) = \varepsilon u_i(t,x) u_j(t,x) + a^2(t,x) h_{ij}(x) \tag{2.1}$$

where a is a function on $\eta(U)$.

Proof. Since V has no zeros, there exist local coordinates $\eta = (t, x)$ such that $V = \partial_t$, hence $V^i = u^i = \delta^i_{\ 0}$ [23, p. 30]. We may assume that $\eta(U)$ is of the form $I \times U_0$ where $I \subset \mathbb{R}$ is an intervall with $0 \in I$. In this comoving coordinate system, $\varepsilon = g_{ij}u^i u^j = g_{00}$ and $u_i = g_{ki}u^k = g_{0i}$. The coordinates of the covariant derivative of u are thus given by

$$\begin{aligned}\nabla_i u_j &= \partial_i g_{0j} - \frac{1}{2} g^{kl} g_{0k}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\ &= \frac{1}{2}(\partial_i g_{0j} - \partial_j g_{0i} + \partial_0 g_{ij}).\end{aligned}$$

For the acceleration, one has in particular:

$$\dot{u}_i = u^k \nabla_k u_i = \nabla_0 u_i = \partial_0 g_{0i} - \frac{1}{2}\partial_i g_{00} = \partial_0 g_{0i}.$$

In the same manner, we obtain a similar expression for the shear:

$$\begin{aligned}\sigma_{ij} &= \frac{1}{2}(\partial_0 g_{ij} + \varepsilon g_{0j}\partial_0 g_{0i} + \varepsilon g_{0i}\partial_0 g_{0j}) - \frac{1}{d-1}\Theta P_{ij} \\ &= \frac{1}{2}\partial_0 P_{ij} - \frac{1}{d-1}\Theta P_{ij}.\end{aligned}$$

Now suppose that $\sigma_{ij} = 0$. We can solve the last equation for the components of the projection tensor P_{ij}, which leads to

$$P_{ij}(t,x) = a^2(t,x) P_{ij}(0,x). \tag{2.2}$$

with

$$a(t,x) = \exp\left(\frac{1}{d-1}\int_0^t \Theta(t',x)\,\mathrm{d}t'\right). \tag{2.3}$$

Defining $h_{ij}(x) = P_{ij}(0,x)$ yields the result. □

Remark 2.2. Shear-free metrics with vanishing acceleration as well as vanishing expansion, i.e., with $a(t,x) = 1$ and $\dot{u}_i(t,x) = 0$, have been studied in [13, 15], called Gödel-type metrics.

3. "Tilted" Twisted Products

Setting $b_i := t_{,i} - \varepsilon u_i$ in the local expression (2.1) for shear-free manifolds, we see that the observer field is g-perpendicular to b: $u^k b_k = \delta^k{}_0 \delta^0{}_k - \varepsilon \cdot u^k u_k = 0$. This motivates the definition of the following geometric structure:

Definition 3.1. Let $\varepsilon \in \{-1, +1\}$. We define the *tilted twisted product* of an interval $(I, \varepsilon \mathrm{d}t^2)$ with a connected Riemannian manifold (N, h) to be the product manifold $M = I \times N$, furnished with the metric tensor

$$g = \varepsilon(\pi_1^* \mathrm{d}t - b)^2 + a^2 \pi_2^* h, \tag{3.1}$$

where a is a given real-valued, positive function on M, and b is a one-form on M which is tangent to each hypersurface $N_s := \{s\} \times N$, i.e., $b(\partial_t) = 0$ where ∂_t is understood as the lift of the canonical vector field on I.

We will also write $(M,g) = (I, \varepsilon \mathrm{d}t^2) \times_{(b,a)} (N,h)$ for this structure. We have that (M,g) is a connected Riemannian (Lorentzian) manifold for $\varepsilon = +1$ ($\varepsilon = -1$) with shear-free reference frame $V = \partial_t$. Any metric admitting a shear-free reference frame is locally isometric to a tilted twisted product. If the twisting function a only depends on t, we will speak of a *tilted warped product*. To keep the notation clear, we will omit the projections π_1, π_2 in the following. Of course, the terminology refers to twisted products [24] and tilted cosmological models [7].

In order to analyze the causal structure and calculate the Ricci curvature of tilted warped products, we need the following:

Lemma 3.2. *The inverse metric of a tilted twisted product* $(M,g) = (I, \varepsilon \mathrm{d}t^2) \times_{(b,a)} (N,h)$ *is given by*

$$g^{-1} = \frac{1}{a^2}\left(h^{-1} + (\|b\|^2 + \varepsilon a^2)\partial_t \otimes \partial_t + \partial_t \vee (h^{-1}(b,\cdot))\right)$$

where h^{-1} *is understood as the lift of the inverse metric on* N*, and* $\|\cdot\|$ *is the norm induced by the Riemannian metric* h*, i.e.,* $\|b\| = (h^{-1}(b,b))^{\frac{1}{2}}$*.*

Proof. Given adapted coordinates on $I \times N$, the metric tensor has the following block matrix form:

$$(g_{ij}) = \begin{pmatrix} \varepsilon & -\varepsilon b \\ -\varepsilon b^T & a^2 h + \varepsilon b^T b \end{pmatrix},$$

with its inverse

$$(g^{ij}) = \frac{1}{a^2}\begin{pmatrix} b h^{-1} b^T + \varepsilon a^2 & b h^{-1} \\ h^{-1} b^T & h^{-1} \end{pmatrix}.$$

□

In adapted coordinates $(t,x) = (t, x_1, \ldots, x_{d-1})$, we have

$$g^{ij} = \frac{1}{a^2}\left(\bar{h}^{ij} + (\|b\|^2 + \varepsilon a^2) u^i u^j + u^i \bar{b}^j + u^j \bar{b}^i\right) \tag{3.2}$$

with $\bar{h}^{ik} h_{kj} = \delta^i{}_j - \delta^0{}_j \delta^i{}_0 = \delta^i{}_j - t_j u^i$ and $\bar{b}^j = \bar{h}^{jk} b_k = -\varepsilon \bar{h}^{jk} u_k$. Any time an index is raised with $\bar{h}^{ij}$, we will indicate this with an overbar.

Causal Structure.

Proposition 3.3. *A tilted twisted product spacetime* $(I, -\mathrm{d}t^2) \times_{(b,a)} (N,h)$ *is stably causal if* $\|b\| < a$*. It is non-chronological if* N *is compact and there exists a slice* N_s *with* $\|b\|_p > a(p)$ *for all* $p \in N_s$*.*

Proof. If $\|b\| < a$, the global coordinate t is a temporal function [2]:

$$g^{-1}(\mathrm{d}t, \mathrm{d}t) = \frac{\|b\|^2 - a^2}{a^2} < 0.$$

It is not difficult to see that in the Lorentzian case, at some point $p \in N_s$, the hyperplane tangent to N_s is spacelike iff $\|b_p\| < a(p)$, timelike iff $\|b_p\| > a(p)$ and lightlike iff $\|b_p\| = a(p)$. If $\|b\| > a$ on a slice, it is a compact timelike hypersurface which must contain a closed timelike curve. (This may only happen if N has vanishing Euler characteristic.) $\square$

Example 3.4. The non-chronological Gödel spacetime [12] with metric

$$g = -\mathrm{d}t^2 - \mathrm{e}^{\sqrt{2}\omega_0}\mathrm{d}t \vee \mathrm{d}y + \mathrm{d}x^2 - \frac{1}{2}\mathrm{e}^{2\sqrt{2}\omega_0 x}\mathrm{d}y^2 + \mathrm{d}z^2$$

defined on $M = \mathbb{R} \times \mathbb{R}^3$ is a tilted product $(M, g) = (\mathbb{R}, -\mathrm{d}t^2) \times_{(b,a)} (\mathbb{R}^3, h)$ with $h = \mathrm{d}x^2 + \frac{1}{2}\mathrm{e}^{2\sqrt{2}\omega_0 x}\mathrm{d}y^2 + \mathrm{d}z^2$ and $b = -\mathrm{e}^{\sqrt{2}\omega_0 x}\mathrm{d}y$. It holds $\|b\|^2 = 2 > 1 = a^2$.

Remark 3.5. Tilted twisted product spacetimes with $a = 1$ and $\|b\| < 1$ are conformally equivalent to *standard stationary spacetimes*, the causal structure of which has recently been studied in a number of articles [8, 10, 19].

Ricci Curvature. The $1 + (d-1)$-composition of the Ricci and scalar curvature of a tilted *warped* product (i.e., a tilted product metric (3.1) with a scale parameter that only depends on the coordinate t) is given by the following:

Theorem 3.6. *Consider a tilted warped product $(M, g) = (I, \varepsilon\mathrm{d}t^2) \times_{(b,a)} (N, h)$ and introduce adapted coordinates (t, x). Then, the components of the Ricci tensor of (M, g) are given by:*

$$\begin{aligned}
R_{ij} = {} & r_{ij} - \dot{b}_i\dot{b}_j - \mathrm{D}_{(i}\dot{b}_{j)} - \ddot{b}_{(i}b_{j)} - (d-3)\theta\mathrm{D}_{(i}b_{j)} - (d-5)\theta\dot{b}_{(i}b_{j)} \\
& + \frac{2\varepsilon}{a^2}\left[\bar{\omega}^k{}_j\omega_{ik} + (2\dot{\bar{b}}^k + (d-3)\theta\bar{b}^k)\omega_{k(j}u_{i)} + \bar{b}^k\dot{\omega}_{k(i}u_{j)} + u_{(j}J_{i)}\right] \\
& + \left[\frac{\|\omega\|^2}{a^4} - \frac{\varepsilon}{2a^2}\left((d-3)\theta(\|b\|^2)^\bullet + (\|b\|^2)^{\bullet\bullet} + 2\mathrm{D}_k\dot{\bar{b}}^k\right) - (d-1)(\theta^2 + \dot{\theta})\right]u_iu_j \\
& + \left[(d-3)(\theta^2 - \dot{\theta})\right]b_ib_j - 2\varepsilon(d-2)\dot{\theta}u_{(i}b_{j)} + 2\varepsilon\theta u_{(i}\dot{b}_{j)} \\
& + \left[-\theta^2((d-3)\|b\|^2 + \varepsilon(d-1)a^2) - \dot{\theta}(\|b\|^2 + \varepsilon a^2) - \theta((\|b\|^2)^\bullet + \mathrm{D}_k\bar{b}^k)\right]h_{ij}.
\end{aligned}$$

Here, r_{ij} are the components of the Ricci tensor of (N, h), and D is the Levi–Civita derivative of the transversal metric h. (Both pulled back via the projection, i.e., $r_{i0} = r_{0i} = 0$ and $\mathrm{D}_0 b_i = 0$ is understood.) Furthermore, $\theta := (\log a)^\bullet$ is the Hubble parameter, $\omega_{ij} = -\varepsilon\mathrm{D}_{[i}b_{j]}$ the vorticity tensor and $J_i := \mathrm{D}_k\bar{\omega}^k{}_i$. The scalar curvature R of (M, g) computes to:

$$\begin{aligned}
a^2R = {} & r - \varepsilon a^{-2}\|\omega\|^2 - (\|b\|^2)^{\bullet\bullet} - (2d-5)\theta(\|b\|^2)^\bullet - 2\mathrm{D}_k\dot{\bar{b}}^k - 2(d-2)\theta\mathrm{D}_k\bar{b}^k \\
& - \varepsilon(d-1)a^2(d\theta^2 + 2\dot{\theta}) - (d-2)\|b\|^2((d-3)\theta^2 + 2\dot{\theta}).
\end{aligned}$$

Proof. If not otherwise noted, indices will be raised and lowered via the metric g, partial derivatives will be denoted by commas. Note that the Christoffel symbols of the Levi–Civita connection induced by the non-tilted, non-warped metric $\bar{g} = \varepsilon\mathrm{d}t^2 + h$ may be written as

$$\bar{\Gamma}^k_{ij} = \frac{1}{2}\bar{h}^{kn}\left(h_{nj,i} + h_{ni,j} - h_{ij,n}\right).$$

Writing $f_{ij} := u_{i,j} - u_{j,i}$, the Christoffel symbols corresponding to g on the other hand are:

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2} g^{kn} \left(g_{nj,i} + g_{ni,j} - g_{ij,n} \right) \\
&= \frac{\varepsilon}{2} \left(u^k (u_{j,i} + u_{i,j}) + u_j f^k{}_i + u_i f^k{}_j \right) \\
&\quad + \frac{1}{2} a^2 g^{kn} (h_{nj,i} + h_{ni,j} - h_{ij,n}) + a g^{kn} (a_{,i} h_{nj} + a_{,j} h_{ni} - a_{,n} h_{ij}).
\end{aligned}$$

Using the equation (3.2) for the inverse metric and plugging

$$\frac{1}{2} a^2 g^{kn} (h_{nj,i} + h_{ni,j} - h_{ij,n}) = \bar{\Gamma}^k_{ij} - \varepsilon u^k u_n \bar{\Gamma}^n_{ij}$$

into this expression gives the following, denoting the Levi–Civita derivative with respect to $\bar{g}$ by a vertical bar:

$$\Gamma^k_{ij} = \bar{\Gamma}^k_{ij} + \frac{\varepsilon}{2} \left((u_{i|j} + u_{j|i}) u^k + u_i f^k{}_j + u_j f^k{}_i \right) + a g^{kn} (a_{,i} h_{nj} + a_{,j} h_{ni} - a_{,n} h_{ij}).$$

Note that $t_{k|i} = 0$ (thus $u_{i|j} = -\varepsilon b_{i|j}$) and $u^k{}_{|i} = 0$. Since we assume that a is constant on the slices, we have $a_{,k} = \dot{a} t_k$ and may write

$$\Gamma^k_{ij} = \bar{\Gamma}^k_{ij} + \beta^k{}_{ij} + (\log a)^{\bullet} \Omega^k{}_{ij} \tag{3.3}$$

where

$$\begin{aligned}
\beta^k{}_{ij} &:= \frac{\varepsilon}{2} \left((u_{i|j} + u_{j|i}) u^k + u_i f^k{}_j + u_j f^k{}_i \right) \\
\Omega^k{}_{ij} &:= a^2 g^{kn} (t_i h_{nj} + t_j h_{ni} - t_n h_{ij}).
\end{aligned}$$

These tensors may be contracted as follows:

$$2\varepsilon \beta^k{}_{kj} = (u_{j|k} + u_{k|j}) u^k + u^k (u_{k|j} - u_{j|k}) = 0 \tag{3.4}$$

and

$$\Omega^k{}_{kj} = a^2 (t^n h_{nj} + t_j h^k{}_k - t^k h_{kj}) = t_j (\delta^k{}_k - \varepsilon u^k u_k) = (d-1) t_j. \tag{3.5}$$

Now the general formula for the components of the Ricci tensor in a given coordinate basis is

$$R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ki,j} + \Gamma^k_{km} \Gamma^m_{ij} - \Gamma^k_{im} \Gamma^m_{kj}.$$

If we plug in (3.3) and account for (3.4), (3.5) the result is:

$$\begin{aligned}
R_{ij} &= \bar{R}_{ij} + \beta^k{}_{ij|k} - \beta^k{}_{im} \beta^m{}_{kj} + ((\log a)^{\bullet})^2 \left((d-1) t_k \Omega^k{}_{ij} - \Omega^k{}_{im} \Omega^m{}_{kj} \right) \\
&\quad + (\log a)^{\bullet\bullet} \left(t_k \Omega^k{}_{ij} - (d-1) t_i t_j \right) \\
&\quad + (\log a)^{\bullet} \left(\Omega^k{}_{ij|k} + (d-1) t_k \beta^k{}_{ij} - \beta^k{}_{im} \Omega^m{}_{kj} - \beta^m{}_{kj} \Omega^k{}_{im} \right).
\end{aligned} \tag{3.6}$$

We will compute the missing terms. First, denoting $\dot u_i = u^k\nabla_k u_i = -\varepsilon u^k b_{i|k} =: -\varepsilon \dot b_i$,

$$\begin{aligned}2\varepsilon\beta^k{}_{ij|k} &= \dot u_{j|i} + \dot u_{i|j} + u_{i|k}f^k{}_j + u_i f^k{}_{j|k} + u_{j|k}f^k{}_i + u_j f^k{}_{i|k}\\ &= f_{ik}f^k{}_j + u_i f^k{}_{j|k} + u_j f^k{}_{i|k} + \dot u_{j|i} + \dot u_{i|j} + \frac{1}{2}\Big((u_{i|k}+u_{k|i})f^k{}_j + (u_{j|k}+u_{k|j})f^k{}_i\Big)\end{aligned}$$

and

$$\begin{aligned}4\beta^k{}_{im}\beta^m{}_{jk} &= \big[(u_{i|m}+u_{m|i})u^k + u_m f^k{}_i + u_i f^k{}_m\big]\cdot\big[(u_{j|k}+u_{k|j})u^m + u_j f^m{}_k + u_k f^m{}_j\big]\\ &= 2\dot u_i\dot u_j - f^2 u_i u_j + 2\dot u^m(u_{i|m}u_j + u_{j|m}u_i)\\ &\quad + \varepsilon f^m{}_j(u_{i|m}+u_{m|i}) + \varepsilon f^m{}_i(u_{j|m}+u_{m|j})\end{aligned}$$

with $f^2 := f_{mn}f^{mn}$. Thus,

$$\begin{aligned}\beta^k{}_{ij|k} - \beta^k{}_{im}\beta^m{}_{jk} &= \frac{\varepsilon}{2}(f_{ik}f^k{}_j + u_i f^k{}_{j|k} + u_j f^k{}_{i|k} + \dot u^m(b_{i|m}u_j + b_{j|m}u_i))\\ &\quad - \frac{1}{2}(\dot b_{j|i} + \dot b_{i|j}) - \frac{1}{2}\dot b_i\dot b_j + \frac{1}{4}f^2 u_i u_j.\end{aligned}\tag{3.7}$$

Note that $2\omega_{ij} = a^4 h_i{}^k h_j{}^l f_{kl}$, and we have $f_{ij} = 2\omega_{ij} + u_i\dot b_j - u_j\dot b_i$ (in particular, $u^n f_{ni} = \varepsilon\dot b_i$) and

$$\begin{aligned}a^2 f^k{}_i &= (\bar h^{kn} + (\|b\|^2+\varepsilon a^2)u^k u^n + u^k\bar b^n + u^n\bar b^k)f_{ni}\\ &= \bar h^{kn}(2\omega_{ni} + u_n\dot b_i - u_i\dot b_n) + \varepsilon(\|b\|^2+\varepsilon a^2)u^k\dot b_i + u^k\bar b^n(2\omega_{ni} + u_n\dot b_i - u_i\dot b_n) + \varepsilon\bar b^k\dot b_i\\ &= 2\bar h^{kn}\omega_{ni} + 2u^k\bar b^n\omega_{ni} + a^2u^k\dot b_i - \bar b^n\dot b_n u^k u_i - \dot{\bar b}^k u_i.\end{aligned}$$

Via similar calculations, this implies with $\|\omega\|^2 := \bar h^{im}\bar h^{jn}\omega_{ij}\omega_{mn}$:

$$a^2 f_{ik}f^k{}_j = 4\bar h^{kn}\omega_{nj}\omega_{ik} + 2\dot{\bar b}^n(\omega_{nj}u_i + \omega_{ni}u_j) - \|\dot b\|^2u_iu_j - \varepsilon a^2\dot b_i\dot b_j\tag{3.8}$$

and

$$f^2 = \frac{4}{a^4}\|\omega\|^2 + \frac{2\varepsilon}{a^2}\|\dot b\|^2.\tag{3.9}$$

Furthermore note with

$$t_k f^k{}_i = \frac{1}{a^2}(2\bar b^n\omega_{ni} - (\bar b^n\dot b_n)u_i) + \dot b_i,\tag{3.10}$$

we have

$$\begin{aligned}a^2 f^k{}_{i|k} &= -2a^2(\log a)^\bullet t_k f^k{}_i + \Big[2\bar h^{kn}\omega_{ni} + 2u^k\bar b^n\omega_{ni} + a^2u^k\dot b_i - \bar b^n\dot b_n u^k u_i - \dot{\bar b}^k u_i\Big]_{|k}\\ &= -4(\log a)^\bullet\bar b^n\omega_{ni} + 2(\log a)^\bullet(\bar b^n\dot b_n)u_i + 2\bar h^{kn}\omega_{ni|k} + 2\bar b^n\dot\omega_{ni} + 2\dot{\bar b}^n\omega_{ni}\\ &\quad + a^2\ddot b_i - (\bar b^n\dot b_n)^\bullet u_i + \varepsilon(\bar b^n\dot b_n)\dot b_i - (\dot{\bar b}^k{}_{|k})u_i + \varepsilon\dot{\bar b}^k b_{i|k}.\end{aligned}\tag{3.11}$$

Also,

$$\begin{aligned}a^2\dot u^k(b_{i|k}u_j + b_{j|k}u_i) &= -\varepsilon(\bar h^{kn} + (\|b\|^2+\varepsilon a^2)u^k u^n + u^k\bar b^n + u^n\bar b^k)(b_{i|k}u_j + b_{j|k}u_i)\dot b_n\\ &= -\varepsilon\dot{\bar b}^k(b_{i|k}u_j + b_{j|k}u_i) - \varepsilon(\bar b^n\dot b_n)(\dot b_iu_j + \dot b_ju_i).\end{aligned}\tag{3.12}$$

Plugging equations (3.8)–(3.12) into (3.7), we get with $\bar\omega^i{}_j := \bar h^{in}\omega_{nj}$:

$$\begin{aligned}\beta^k{}_{ij|k} - \beta^k{}_{im}\beta^m{}_{jk} = {} & -\dot b_i \dot b_j - \frac{1}{2}(\dot b_{i|j} + \dot b_{j|i}) + \frac{\varepsilon}{2}(\ddot b_i u_j + \ddot b_j u_i) + \frac{\|\omega\|^2}{a^4} u_i u_j \\ & + \frac{\varepsilon}{a^2}\Big[2\bar\omega^k{}_j \omega_{ik} + (2(\log a)^{\bullet}\bar b^n \dot b_n - (\bar b^n \dot b_n)^{\bullet} - \dot{\bar b}^k{}_{|k})u_i u_j \\ & + 2(\dot{\bar b}^n - (\log a)^{\bullet}\bar b^n)(\omega_{nj}u_i + \omega_{ni}u_j) + \bar b^n(\dot\omega_{ni}u_j + \dot\omega_{nj}u_i) \\ & + \bar\omega^k{}_{i|k}u_j + \bar\omega^k{}_{j|k}u_i\Big] \end{aligned} \tag{3.13}$$

The following formulas may be easily found by realizing that $b_i = a^2 t_k h^k{}_i = a^2 b_k h^k{}_i$:

$$(d-1)t_k\Omega^k{}_{ij} - \Omega^k{}_{im}\Omega^m{}_{jk} = (d-3)b_i b_j - (d-1)u_i u_j - (d-3)(\|b\|^2 + \varepsilon a^2)h_{ij}, \tag{3.14}$$

$$t_k\Omega^k{}_{ij} - (d-1)t_i t_j = -(d-3)b_i b_j - (d-1)u_i u_j - (\|b\|^2 + \varepsilon a^2)h_{ij} - \varepsilon(d-2)(u_i b_j + b_i u_j). \tag{3.15}$$

As for the remaining terms, those compute to:

$$\Omega^k{}_{ij|k} = \dot b_i b_j + b_j \dot b_i + \varepsilon(u_i \dot b_j + \dot b_i u_j) - \left(2\bar b^n \dot b_n + 2\varepsilon a\dot a + \bar b^k{}_{|k}\right) h_{ij}, \tag{3.16}$$

$$t_m\beta^m{}_{ij} = -\frac{1}{2}(b_{i|j} + b_{j|i}) + \frac{\varepsilon}{a^2}\bar b^n(\omega_{ni}u_j + \omega_{nj}u_i) + \frac{\varepsilon}{2}(\dot b_i u_j + u_i \dot b_j) - \frac{\varepsilon}{a^2}(\bar b^n \dot b_n)u_i u_j, \tag{3.17}$$

$$\beta^m{}_{ik}\Omega^k{}_{mj} = -b_{j|i}. \tag{3.18}$$

We note that $\bar R_{ij} = r_{ij}$ and $b_{i|j} = D_j b_i + \dot b_i t_j$. With this, plugging (3.13)–(3.18) into (3.6) finally yields the expression for the Ricci tensor; the scalar curvature of course follows from computing the trace (with respect to g). $\square$

For the remainder of this work, we will be concerned with applications in general relativity, and the following will be useful:

Corollary 3.7. *Suppose Einstein's equations $R_{ij} - \frac{R}{2}g_{ij} = \tau_{ij}$ hold for a tilted warped product spacetime $(I, -\mathrm{d}t^2) \times_{(b,a)} (N, h)$ with $d = 4$. Let the usual $1+3$-decomposition of the energy–momentum tensor be given by*

$$\tau_{ij} = \rho u_i u_j + p^* P_{ij} + 2q_{(i}u_{j)} + \Pi_{ij},$$

with energy density ρ, *(effective) pressure* p^*, *heat flux* q_i *and anisotropic pressure* Π_{ij}. *Then,*

$$\begin{aligned}
a^2\rho &= \frac{r}{2} + \frac{3\|\omega\|^2}{2a^2} - \theta(\|b\|^2)^{\bullet} - 2\theta \mathrm{D}_k\bar{b}^k + 3a^2\theta^2 - \|b\|^2(\theta^2 + 2\dot{\theta}),\\
3a^2p^* &= -\frac{r}{2} + \frac{\|\omega\|^2}{2a^2} + 2\theta(\|b\|^2)^{\bullet} + (\|b\|^2)^{\bullet\bullet} + 2\mathrm{D}_k\dot{\bar{b}}^k\\
&\quad + 2\theta D_k\bar{b}^k - 9a^2\theta^2 - 6a^2\dot{\theta} + \|b\|^2(\theta^2 + 2\dot{\theta}),\\
q_i &= -\frac{2}{a^2}\left((2\dot{\bar{b}}^k + \theta\bar{b}^k)\omega_{ki} + \bar{b}^k\dot{\omega}_{ki} + J_i\right) + 4\dot{\theta}b_i - 2\theta\dot{b}_i,\\
\Pi_{ij} &= r_{ij} - \dot{b}_i\dot{b}_j - \mathrm{D}_{(i}\dot{b}_{j)} - \ddot{b}_{(i}b_{j)} - \theta\mathrm{D}_{(i}b_{j)} + \theta\dot{b}_{(i}b_{j)}\\
&\quad - \frac{2}{a^2}\bar{\omega}^k{}_j\omega_{ik} + (\theta^2 - \dot{\theta})b_ib_j - \alpha h_{ij},
\end{aligned}$$

where the function α *is determined by the condition* $\bar{h}^{ij}\Pi_{ij} = 0$.

Proof. This is a straightforward computation using the formulas

$$\rho = R_{kl}u^ku^l + \frac{R}{2},\ 3p^* = R_{kl}u^ku^l - \frac{R}{2},$$
$$q_i = a^2h^k{}_iR_{kl}u^l,\ \Pi_{ij} = a^4h^k{}_ih^l{}_jR_{kl} - (\dots)h_{ij}.$$

□

4. The Tilted Closed Robertson–Walker Model

As an application of tilted warped products and the Ricci and scalar curvature formulas stated in Theorem 3.6, in this section we will provide an explicit example of a conformally stationary tilted warped product spacetime. For another example of a closed rotating and expanding world model, see [5].

First, we note the following general fact:

Proposition 4.1. *The standard reference frame* $V = \partial_t$ *of a tilted warped product is parallel to a local conformal vector field iff the shift is of the form*

$$b_j(t,x) = a(t)\cdot\left(\chi_j(x) + \int_0^t \frac{\mathrm{d}\tau}{a(\tau)}\cdot\xi_j(x)\right),$$

where $\xi_j(x)$ *are the components of a closed one-form defined on the fiber* N*:* $\mathrm{D}_{[i}\xi_{j]} = 0$.

Proof. It is well-known that a shear-free reference frame V with $u = g(V,\cdot)$ is parallel to a local conformal vector field iff the one-form $\dot{u} - \theta u$ is closed, cf. for example [16]. For tilted product spacetimes this translates to the set of equations

$$\ddot{b}_j = (\theta b_j)^{\bullet},\quad \mathrm{D}_{[i}\dot{b}_{j]} = \theta\mathrm{D}_{[i}b_{j]}.$$

The first equation may be readily integrated to give the desired expression for $b_j(t,x)$ whereas the second yields the constraint for $\xi_j(x)$. □

The proposition holds globally if the fiber N and therefore $M = I \times N$ is simply-connected. Thus, examples of tilted warped product spacetimes with conformally stationary reference frame may be easily constructed. Such examples are of importance since these models are parallax-free [16]. One such example is given in the following: Set up coordinates (r, μ, ν) on $S^3 \subset \mathbb{R}^4$ defining the parametrization [29]:

$$\begin{aligned}&[-1,1] \times [0, 2\pi]^2 \to S^3,\\ &(r, \mu, \nu) \mapsto (r\cos\mu, r\sin\mu, \sqrt{1-r^2}\cos\nu, \sqrt{1-r^2}\sin\nu).\end{aligned}$$

Now consider the manifold $M = I \times S^3$ with a suitable interval $I \subset \mathbb{R}$, equipped with the tilted warped product metric

$$g = -(\mathrm{d}t - b_0 a(r^2\mathrm{d}\mu + (1-r^2)\mathrm{d}\nu)^2 + a^2(\frac{1}{1-r^2}\mathrm{d}r^2 + r^2\mathrm{d}\mu^2 + (1-r^2)\mathrm{d}\nu^2),$$

where $b_0 \in \mathbb{R}$ and a is a function defined on I. The spatial metric h is just the canonical metric on S^3. Provided that $b_0 \neq 0$, the shift one-form b does not vanish anywhere and is in fact, for each fixed $s \in I$, h-metrically equivalent to the left-invariant vector field on S^3, $\bar{b} = b_0 a(s)(\partial_\mu + \partial_\nu)$.

The spacetime (M, g) generalizes the closed Robertson–Walker spacetime. We have $\|b\|^2 = b_0^2 a^2$, and (M, g) is stably causal for $|b_0| < 1$ by virtue of Prop. 3.3. Furthermore, the fiber S^3 is compact, and the spacetime is thus totally vicious for $|b_0| > 1$ (i.e., contains a closed timelike curve through every point) since the observer field $V = \partial_t$ is conformally stationary by Prop. 4.1 (cf. [20, p. 113]).

We will follow the rationale in [6] and assume the energy–momentum tensor to be of a viscous fluid form:

$$\tau = \rho u \otimes u + (p - \zeta\Theta)P + q \vee u - 2\eta\sigma, \tag{4.1}$$

where $\zeta, \eta \geq 0$ are functions representing the bulk and shear viscosity coefficients, respectively.

Since the model is shear-free, the form of the energy–momentum tensor (4.1) enforces that the anisotropic pressure vanishes. Since h is an Einstein metric (i.e., $r_{ij} = \frac{r}{3}h_{ij}$), and $\bar{b}$ is a Killing vector field on the slices (i.e., $\mathrm{D}_{(i}b_{j)} = 0$), it may be readily verified via Corollary 3.7 that this condition is equivalent to

$$b_i b_j(-\ddot{a}a + (\dot{a})^2) - \bar{\omega}^k{}_j\omega_{ik} = \frac{1}{3}(\|b\|^2(-\ddot{a}a + (\dot{a})^2) + \|\omega\|^2)h_{ij}.$$

Noting that $\omega_{ij} = \mathrm{D}_{[i}b_{j]}$, this equation may be evaluated in a straightforward manner. One then sees that in the case $b_0 \neq 0$ the anisotropic pressure vanishes iff the scale factor satisfies the differential equation

$$\ddot{a}a - (\dot{a})^2 + 1 = 0. \tag{4.2}$$

Note that the standard Robertson–Walker model (i.e., the case $b_0 = 0$) is shear-free for any scale parameter. Introducing vorticity in the way done here thus yields a tight constraint that seriously limits cosmic evolution and matter models.

Using this equation to eliminate derivatives of a of order larger than one, the remaining components of the energy–momentum tensor are:

$$\begin{aligned}\rho &= a^{-2}\left(-3(\dot{a})^2(b_0^2-1)+5b_0^2+3\right),\\ p-\zeta\Theta &= a^{-2}\left(3(\dot{a})^2(b_0^2-1)-b_0^2+1\right),\\ q &= -4a^{-2}b.\end{aligned}$$

Note in particular that in case of a vanishing bulk viscosity coefficient, $\zeta=0$, the fluid is barotopic with equation of state

$$\rho = -p + 4a^{-2}(b_0^2+1).$$

We would also like to note that the model is purely Weyl electric.
As done in [6], we will in the following stipulate the thermodynamical conditions:

$$\operatorname{div}(n\cdot V)=0, \quad \text{particle number conservation law,} \tag{4.3}$$

$$\mathrm{d}\left(\frac{S}{n}\right)=\frac{1}{T}\mathrm{d}\left(\frac{\rho}{n}\right)+\frac{p}{T}\mathrm{d}\left(\frac{1}{n}\right), \quad \text{Gibbs' relation,} \tag{4.4}$$

$$q=-kP\left(\nabla T+T\nabla_V V,\,\cdot\,\right), \quad \text{temperature gradient law,} \tag{4.5}$$

where $n,S,k,T\geq 0$ are the particle number density, entropy density, thermal conductivity and temperature, respectively. These conditions are designed so that the second law of thermodynamics, $\operatorname{div}(SV+\frac{1}{T}q^\sharp)\geq 0$, holds. Note, however, that more recent treatments of relativistic thermodynamics suggest that these constraints are too simple to give a full account of the actual physics. For an overview of different approaches of the problem, cf. for example [1]. Like in the non-tilted case, particle conservation (4.3) enforces $n\propto a^{-3}$.

Vanishing Bulk Viscosity. For simplicity and to provide a more concrete example, we will also make the following additional assumptions:

1. Each of the thermodynamical variables only depends on the "cosmological development parameter" t (remember that t is only a temporal function for $|b_0|<1$),
2. bulk viscosity effects are negligible, i.e., $\zeta=0$,
3. the entropy density satisfies the equation of state $S=\frac{\rho+p}{T}$ (cf. [1, Sec. 14.1] with vanishing diffusion).

Accounting for (4.2), we may solve (4.5) for the thermal conductivity: $k=\frac{4}{a(aT)^\bullet}$.
With the above constraints, Gibbs' relation (4.4) also simplifies considerably and may be solved with respect to the temperature (T_0 is some positive constant):

$$T=T_0\cdot a^{-\frac{b_0^2-1}{b_0^2+1}}.$$

We may now also state explicit formulas for the entropy density and the thermal conductivity:

$$S = \frac{4}{T_0}(b_0^2+1)a^{-\frac{b_0^2+3}{b_0^2+1}},\ k = \frac{2}{T_0\dot{a}}(b_0^2+1)a^{-\frac{2}{b_0^2+1}}.$$

In order for the thermal conductivity to be well-defined and non-negative, the expansion must be positive: $\dot{a} > 0$. The entropy production is given by

$$\operatorname{div}(SV + \frac{1}{T}q^\sharp) = \frac{8b_0^2}{T_0(b_0^2+1)}\dot{a}a^{-\frac{2(b_0^2+2)}{b_0^2+1}}.$$

We have yet to determine the scale factor. The isotropic-pressure condition (4.2) may actually be solved analytically via the substitution $a = e^f$, yielding the autonomous equation $\ddot{f} + e^{-2f} = 0$ the integration of which is elementary. The solutions are:

$$\begin{aligned} a_-(t) &= \alpha \sinh\left(\frac{t-t_0}{\alpha}\right), && t_0 < t, \\ a_0(t) &= t - t_0, && t_0 < t, \\ a_+(t) &= \frac{1}{\omega}\sin(\omega(t-t_0)), && t_0 < t < \frac{\pi}{\omega} + t_0. \end{aligned}$$

In any case, the spacetime features a singularity. These three classes of solutions may be characterized by the deceleration parameter, $-\frac{a\ddot{a}}{\dot{a}^2}$, which is non-positive (zero, non-negative) for a_- (a_0, a_+). Note that a_+ violates the condition $\dot{a} > 0$ and $k(t)$ becomes singular as t approaches $t_0 + \frac{\pi}{2\omega}$. With the help of Corollary 1 in [21], it is easy to check that the dominant and strong energy conditions are satisfied for $a = a_0$ and $a = a_+$. In the case $a = a_-$, the strong energy condition is satisfied iff $b_0 \geq 1$ and the dominant and weak energy conditions are satisfied iff $b_0 \leq 1$.

Dust Solutions. We will now drop the constraint $\zeta = 0$ and instead have a look at the viscous dust solutions of the model, i.e., $p = 0$. This condition implies

$$\zeta = \frac{1-b_0^2}{3a\dot{a}}(3(\dot{a})^2 - 1).$$

In order for ζ to be well-defined and non-negative, we must have $a = a_0$ or $a = a_-$, and $|b_0| \leq 1$. Furthermore, the temperature is constant ($T = T_0$), and the other thermodynamical variables compute to:

$$S = \frac{1}{a^2T_0}\left(3(1-b_0^2)(\dot{a})^2 + 3b_0^2 + 5\right),\ k = \frac{4}{T_0a\dot{a}}.$$

Entropy production is given by

$$\operatorname{div}(SV + \frac{1}{T}q^\sharp) = \frac{\dot{a}}{a^3T_0}\left(9(1-b_0^2)(\dot{a})^2 + b_0^2 + 3\right).$$

5. Barotropic Perfect Fluid and Dust Solutions with Geodesic Flow

We now wish to classify tilted warped product spacetimes with geodesic perfect fluid source, i.e., we have $\dot{b}_i = 0$ and

$$R_{ij} - \frac{R}{2} g_{ij} = (\rho + p) u_i u_j + p g_{ij},$$

where $\rho + p \neq 0$ on any open subset. Furthermore, we require the fluid to be barotropic, i.e., an equation of state $\rho = \rho(p)$ with $\frac{\mathrm{d}\rho}{\mathrm{d}p} \neq 0$ holds. Kinematically, models with vanishing acceleration and shear are important since they follow an isotropic Hubble law of first order [17].

Proposition 5.1. *For any (4-dimensional) simply-connected tilted warped product spacetime with barotropic geodesic perfect fluid source, either one of the two following cases may occur:*

1. *The vorticity vanishes. In this case, (M, g) is isometric to a Robertson–Walker spacetime. Furthermore, (N, h) is conformally equivalent to a space of constant curvature.*
2. *The expansion vanishes. In this case, the following holds:*
   (a) *The pressure p, energy density ρ and rotation scalar $\|\omega\|^2$ are constant.*
   (b) *The fiber (N, h) is a space of constant scalar curvature.*
   (c) *The fiber (N, h) is a space of constant curvature iff the vorticity also vanishes iff the spacetime is Einstein's static universe.*

Proof. In the following we will make frequent use of Corollary 3.7. By a Robertson–Walker spacetime we mean a warped product $(\tilde{I} \times N, -\mathrm{d}t^2 + \tilde{h})$ with a fiber $(N, \tilde{h})$ of constant curvature (which in $d - 1 = 3$ dimensions is equivalent to $(N, \tilde{h})$ being Einstein [3, Prop. 1.120]). One may also require the fiber to be complete in which case the spacetime is not only stably causal but globally hyperbolic [25]. Since the shear-free fluid conjecture is known to be true for vanishing acceleration, either the vorticity or the expansion vanishes [26]. Let us first examine the case of vanishing vorticity. In this case, we must have $b = \mathrm{d}f$ with some function $f \colon N \to \mathbb{R}$ (since we assume N to be simply-connected, [23, pp. 358–360]). Furthermore, the heat flux vanishes iff $\dot{\theta} = 0$ or $b_i = 0$. If the shift vanishes, $b_i = 0$, we are finished since the vanishing of the anisotropic pressure immediately yields $r_{ij} = \alpha h_{ij}$. Thus, $a(t) = c\mathrm{e}^{\theta t}$ for some positive constant c. If we pull back the metric

$$g = -(\mathrm{d}t - \mathrm{d}f)^2 + c^2 \mathrm{e}^{2\theta t} h$$

via the diffeomorphism $\Phi(t, x) = (t + f(x), x)$ we see that it becomes a warped product:

$$\Phi^* g = -\mathrm{d}t^2 + c^2 \mathrm{e}^{2\theta t} \mathrm{e}^{2\theta f} h.$$

The fiber metric $\tilde{h} = \mathrm{e}^{2\theta f} h$ is conformally equivalent to h; the Ricci tensor may be computed with the usual formula [3, Theorem 1.159]:

$$\tilde{r}_{ij} = r_{ij} - \theta \mathrm{D}_i f_{,j} + \theta^2 f_{,i} f_{,j} - (\dots) h_{ij} \tag{5.1}$$

$$= r_{ij} - \theta \mathrm{D}_{(i} b_{j)} + \theta^2 b_i b_j - (\dots) h_{ij}. \tag{5.2}$$

Plugging this into the formula for the anisotropic pressure, we have $0 = \Pi_{ij} = \tilde{r}_{ij} - \tilde{\alpha} \tilde{h}_{ij}$ as desired.
For the second case of vanishing expansion ($a = \text{const.}$), the conditions of a vanishing heat flux and anisotropic pressure yield

$$\bar{\rho} = \tfrac{1}{2} r + \tfrac{3}{2} \|\bar{\omega}\|^2, \quad \bar{p} = -\tfrac{1}{6} r + \tfrac{1}{6} \|\bar{\omega}\|^2, \quad \mathrm{D}_k \bar{\omega}^k{}_i = 0, \tag{5.3}$$

$$r_{ij} = 2 \bar{\omega}_i{}^k \bar{\omega}_{kj} + \tfrac{1}{3} \left(2 \|\bar{\omega}\|^2 + r \right) h_{ij}, \tag{5.4}$$

where $\bar{\rho} = a^2 \rho$, $\bar{p} = a^2 p$, $\bar{\omega}_{ij} = a \omega_{ij}$. It is clear from these formulas and momentum conservation ($h^m{}_j p_{,m} = -a^2 (\rho + p) \dot{b}_j = 0$) that the gradient of p vanishes. The equation of state then implies that ρ and therefore r and $\|\omega\|^2$ are constant, too.
For the last claim (c), observe that Eq. (5.4) in an orthonormal basis shows that the 3-geometry satisfies the Einstein condition, i.e., has constant curvature, iff the antisymmetric matrix $(\bar{\omega}_{ij})$ vanishes or is a non-trivial square root of a multiple of the identity matrix; but the latter is not possible in three dimensions. The spacetime is thus a static Robertson–Walker solution. □

Since shear-free dusts also either expand or rotate [9], we may state by very similar arguments:

Proposition 5.2. *For any simply-connected tilted warped product spacetime (M, g) with geodesic dust source, either one of the two following cases may occur:*

(1) *The vorticity vanishes. In this case, (M, g) is a Robertson–Walker dust solution.*
(2) *The expansion vanishes. In this case, the vorticity also vanishes iff the spacetime is flat Minkowski.*

Example 5.3. The Gödel spacetime is an example of a rotating spacetime covered by Proposition 5.1; the van Stockum spacetime [22, 27] is a rotating dust solution covered by Proposition 5.2.

6. Summary and Outlook

Apart from constructing further explicit examples of shear-free cosmological models and investigating other more general classes (for example, rigidly rotating matter), many interesting subjects remain, among which are:

- The causal structure of tilted product spacetimes with $\|b\| = a$, or with $\|b\| > a$ and non-compact fiber (like the classical Gödel spacetime).
- The curvature structure of the general twisting case $a = a(t, x)$.

- Necessary and sufficient conditions for a shear-free spacetime to be *globally* of tilted product type, or conformally equivalent to a tilted product.

Finally, one might also want to investigate the Riemannian case, for which we expect interesting applications as well. For example, shear-free reference frames with vanishing expansion define Riemannian flows [28] with normal bundle metric P.

References

[1] N. Andersson, G. L. Comer, *Relativistic Fluid Dynamics: Physics for Many Different Scales*, Living Rev. Relativity, **10** (2007), 1–83.

[2] A. N. Bernal, M. Sánchez, *Smoothness of Time Functions and the Metric Splitting of Globally Hyperbolic Spacetimes*, Comm. Math. Phys., **257** (2005), 43–50.

[3] A. Besse, *Einstein Manifolds* (Springer, 1987).

[4] T. Chrobok, Y. N. Obukhov, M. Scherfner, *On the Construction of Shear-Free Cosmological Models*, Mod. Phys. Lett. A, **20** (2001), 1321–1325.

[5] T. Chrobok, Y. N. Obukhov, M. Scherfner, *Shear-Free Rotating Inflation*, Phys. Rev. D, **66** (2002), 043518.

[6] A. A. Coley, *On the Thermodynamics of Cosmological Models with Heat Conduction*, Phys. Lett. A, **137** (1989), 235–238.

[7] A. A. Coley, S. Hervik, W. C. Lim, *Fluid Observers and Tilting Cosmology*, Class. Quant. Grav., **23** (2006), 3573–3591.

[8] A. Dirmeier, M. Plaue, M. Scherfner, *Growth Conditions, Riemannian Completeness and Lorentzian Causality*, to appear in J. Geom. Phys.

[9] G. F. R. Ellis, *Dynamics of Pressure-Free Matter in General Relativity*, J. Math. Phys., **29** (1967), 1171–1194.

[10] J. L. Flores, *The Causal Boundary Of Standard Stationary Spacetimes*, AIP Conf. Proc., **1122** (2009), 272–275.

[11] E. García-Río, D. N. Kupeli, *Singularity Versus Splitting Theorems for Stably Causal Spacetimes*, Ann. Global. Analysis Geom., **14** (1996), 301–312.

[12] K. Gödel, *An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation*, Rev. Mod. Phys., **21** (1949), 447–450.

[13] M. Gürses, A. Karasu, O. Sarioglu, *Gödel-type Metrics in Various Dimensions*, Class. Quant. Grav., **22** (2005), 1527–1543.

[14] M. Gürses, M. Plaue, M. Scherfner, T. Schönfeld, L. A. M. de Sousa jr., *On Spacetimes with Given Kinematical Invariants: Construction and Examples*, in: *Gödel-type Spacetimes: History and New Developments* (Kurt Gödel Society, 2010).

[15] M. Gürses, O. Sarioglu, *Gödel-type Metrics in Various Dimensions II: Inclusion of a Dilaton Field*, Class. Quant. Grav., **22** (2005), 4699–4713.

[16] W. Hasse, V. Perlick, *Geometrical and Kinematical Characterization of Parallax-Free World Models*, J. Math. Phys., **29** (1988), 2064–2068.

[17] W. Hasse, V. Perlick, *On Spacetime Models with an Isotropic Hubble Law*, Class. Quant. Grav., **16** (1999), 2559–2576.

[18] S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, 1973).

[19] M. A. Javaloyes, M. Sánchez, *A Note on the Existence of Standard Splittings for Conformally Stationary Spacetimes*, Class. Quant. Grav., **25** (2008), 168001.

[20] P. S. Joshi, *Global Aspects in Gravitation and Cosmology* (Clarendon Press, 1993).

[21] C. A. Kolassis, N. O. Santos, D. Tsoubelis, *Energy Conditions for an Imperfect Fluid*, Class. Quant. Grav., **5** (1988), 1329–1338.

[22] K. Lanczos, *Über eine stationäre Kosmologie im Sinne der Einsteinschen Gravitationstheorie*, Zeitschrift für Physik A: Hadrons and Nuclei, **21** (1923), 73–110.
[23] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity* (Academic Press, 1983).
[24] R. Ponge, H. Reckziegel, *Twisted Products in pseudo-Riemannian Geometry*, Geometriae Dedicata, **48** (1993), 15–25.
[25] M. Sánchez, *On the Geometry of Generalized Robertson–Walker Spacetimes: Curvature and Killing Fields*, J. Geom. Phys., **31** (1999), 1–15.
[26] J. M. M. Senovilla, C. F. Sopuerta, P. Szekeres, *Theorems on Shearfree Perfect Fluids with their Newtonian Analogues*, Gen. Rel. Grav., **30** (1998), 389–411.
[27] W. J. van Stockum, *The Gravitational Field of a Distribution of Particles Rotating Around an Axis of Symmetry*, Proc. Roy. Soc. Edinburgh A, **57** (1937), 135.
[28] P. Tondeur, *Geometry of Foliations* (Birkhäuser, 1997).
[29] F. Wilson, *Some Examples of Vector Fields on the 3-Sphere*, Ann. de l'institut Fourier, **20** (1970), 1–20.

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