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Dynamical systems and Poisson structures

Metin Gürses,[1,a)] Gusein Sh. Guseinov,[2,b)] and Kostyantyn Zheltukhin[3,c)]
[1]*Department of Mathematics, Faculty of Sciences, Bilkent University, 06800 Ankara, Turkey*
[2]*Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey*
[3]*Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey*

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We first consider the Hamiltonian formulation of $n=3$ systems, in general, and show that all dynamical systems in $\mathbb{R}^3$ are locally bi-Hamiltonian. An algorithm is introduced to obtain Poisson structures of a given dynamical system. The construction of the Poisson structures is based on solving an associated first order linear partial differential equations. We find the Poisson structures of a dynamical system recently given by Bender *et al.* [J. Phys. A: Math. Theor. **40**, F793 (2007)]. Secondly, we show that all dynamical systems in $\mathbb{R}^n$ are locally $(n-1)$-Hamiltonian. We give also an algorithm, similar to the case in $\mathbb{R}^3$, to construct a rank two Poisson structure of dynamical systems in $\mathbb{R}^n$. We give a classification of the dynamical systems with respect to the invariant functions of the vector field $\vec{X}$ and show that all autonomous dynamical systems in $\mathbb{R}^n$ are superintegrable. © *2009 American Institute of Physics.* [doi:10.1063/1.3257919]

I. INTRODUCTION

Hamiltonian formulation of $n=3$ systems has been intensively considered in the last two decades. Works[1,2] on this subject give a very large class of solutions of the Jacobi equation for the Poisson matrix J. Recently generalizing the solutions given in Ref. 1 we gave the most general solution of the Jacobi equation in $\mathbb{R}^3$.[3] Matrix $J=(J^{ij})$, $i,j=1,2,\ldots,n$ defines a Poisson structure in $\mathbb{R}^n$ if it is skew symmetric, $J^{ij}=-J^{ji}$, and its entries satisfy the Jacobi equation,

$$J^{li}\partial_l J^{jk} + J^{lj}\partial_l J^{ki} + J^{lk}\partial_l J^{ij} = 0, \tag{1}$$

where $i,j,k=1,2,\ldots,n$. Here we use the summation convention, meaning that repeated indices are summed up. We showed in Ref. 3 that the general solution of the above equation (1) in the case $n=3$ has the form

$$J^{ij} = \mu\epsilon^{ijk}\partial_k\Psi, \quad i,j=1,2,3, \tag{2}$$

where μ and Ψ are arbitrary differentiable functions of x^i,t, $i=1,2,3$, and ϵ^{ijk} is the Levi–Civita symbol. Here t should be considered as a parameter. In the same work we have also considered a bi-Hamiltonian representation of Hamiltonian systems. It turned out that any Hamiltonian system in $\mathbb{R}^3$ has a bi-Hamiltonian representation.

In the present paper we prove that any n-dimensional dynamical system,

$$\dot{\vec{x}} = \vec{X}(x^1,x^2,\ldots,x^n,t), \tag{3}$$

where $\vec{x}=(x^1,x^2,\ldots,x^n)$ is Hamiltonian, that is, has the form

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[a)]Electronic mail: gurses@fen.bilkent.edu.tr.
[b)]Electronic mail: guseinov@atilim.edu.tr.
[c)]Electronic mail: zheltukh@metu.edu.tr.

$$\dot{x}^i = J^{ij}\partial_j H, \quad i=1,2,\ldots,n, \tag{4}$$

where $J=(J^{ij})$ is a Poisson matrix and H, as well as J^{ij}, are differentiable functions of the variables $x_1,x_2,\ldots,x_n,t$. Moreover, we show that the system (3) is $(n-1)$-Hamiltonian. This problem in the case $n=3$ was considered in Refs. 4 and 5, where authors start with an invariant of the dynamical system as a Hamiltonian and then proceed by writing the system in the form (4) and imposing conditions on J so that it satisfies the Jacobi equation. But proofs given in these works are, as it seems to us, incomplete and not satisfactory.

Using (2) for matrix J we can write Eq. (4) in $\mathbb{R}^3$ as

$$\dot{\vec{x}} = \mu \vec{\nabla}\Psi \times \vec{\nabla} H. \tag{5}$$

Let $\vec{X}$ be a vector field in $\mathbb{R}^3$. If H_1 and H_2 are two invariant functions of $\vec{X}$, i.e., $\vec{X}(H_\alpha) = X^j\partial_j H_\alpha = 0$, $\alpha=1,2$, then $\vec{X}$ is parallel to $\vec{\nabla}H_1 \times \vec{\nabla}H_2$. Therefore

$$\vec{X} = \mu \vec{\nabla}H_1 \times \vec{\nabla}H_2, \tag{6}$$

where the function μ is a coefficient of proportionality. The right-hand side of Eq. (6) is in the same form as the right-hand side of Eq. (5), so $\vec{X}$ is a Hamiltonian vector field. We note that the equation which allows to find the invariants of a vector field $\vec{X}$ is a first order linear partial differential equation. We remark here that dynamical systems in $\mathbb{R}^3$ differ from the dynamical systems in $\mathbb{R}^n$ for $n>3$. We know the general solution (2) of the Jacobi equation (1) in $\mathbb{R}^3$. In $\mathbb{R}^n$, as we shall see in Sec. III, we know only the rank 2 solutions of the Jacobi equations for all n.

An important difference of our work, contrary to other works in the subject, is that in the construction of the Poisson structures we take into account the invariant functions of the vector field $\vec{X}$ rather than the invariants (constants of motion) of the dynamical system. The total time derivative of a differentiable function F in $\mathbb{R}^n$ along the phase trajectory is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \vec{X}\cdot\vec{\nabla}F. \tag{7}$$

An invariant function of the vector field $\vec{X}(x^1,x^2,\ldots,x^n,t)$, i.e., $\vec{X}\cdot\vec{\nabla}F=0$, is not necessarily an invariant function (constant of motion) of the dynamical system. For autonomous systems where $\vec{X}=\vec{X}(x^1,x^2,\ldots,x^n)$ these invariant functions are the same. We give a representation of the vector field $\vec{X}$ in terms of its invariant functions. We show that all autonomous dynamical systems are superintegrable. A key role plays the existence of $n-1$ functionally independent solutions $\zeta_\alpha(x^1,x^2,\ldots,x^n,t)$ $(\alpha=1,2,\ldots,n-1)$ of the linear partial differential equation,

$$\vec{X}\cdot\vec{\nabla}\zeta \equiv X^1\frac{\partial \zeta}{\partial x^1} + X^2\frac{\partial \zeta}{\partial x^2} + \cdots + X^n\frac{\partial \zeta}{\partial x^n} = 0, \tag{8}$$

where $X^i=X^i\ (x^1,x^2,\ldots,x^n,t)$, $i=1,2,\ldots,n$, are given functions (see Refs. 6–8). For all $\alpha=1,2,\ldots,n-1$, $\vec{\nabla}\zeta_\alpha$ is perpendicular to the vector field $\vec{X}$. This leads to the construction of the rank 2 Poisson tensors for $n>3$,

$$J_\alpha^{ij} = \mu\epsilon^{\alpha\alpha_1\alpha_2\cdots\alpha_{n-2}}\epsilon^{ijj_1\cdots j_{n-2}}\partial_{j_1}\zeta_{\alpha_1}\partial_{j_2}\zeta_{\alpha_2}\cdots\partial_{j_{n-2}}\zeta_{\alpha_{n-2}}, \tag{9}$$

where $i,j=1,2,\ldots,n$ and $\alpha=1,2,\ldots,n-1$. Here $\epsilon^{ijj_1\cdots j_{n-2}}$ and $\epsilon^{\alpha\alpha_1\alpha_2\cdots\alpha_{n-2}}$ are Levi–Civita symbols in n and $n-1$ dimensions, respectively. Any dynamical system with the vector field $\vec{X}$ possesses Poisson structures in the form given in (9). Hence we can give a classification of dynamical systems in $\mathbb{R}^n$ with respect to the invariant functions of the vector field $\vec{X}$. There are mainly three classes where the superintegrable dynamical systems constitute the first class. By the use of the invariant functions of the vector field $\vec{X}(x^1,x^2,\ldots,x^n,t)$, in general, we give a Poisson structure in $\mathbb{R}^n$ which has rank 2. For autonomous systems, the form (9) of the above Poisson structure first was given in Refs. 9 and 10.

Our results in this work are mainly local. This means that our results are valid in an open domain of $\mathbb{R}^n$ where the Poisson structures are different from zero. In Ref. 3 we showed that the Poisson structure (2) in $\mathbb{R}^3$ preserves its form in the neighborhood of irregular points, lines, and planes. Note also that our construction of the Poisson structures is explicit if we can solve (8) explicitly.

In Sec. II we give new proofs of the formula (2) and prove that any dynamical system in $\mathbb{R}^3$ is Hamiltonian. So, following Ref. 3 we show that any dynamical system in $\mathbb{R}^3$ is bi-Hamiltonian. Applications of these theorems to several dynamical systems are presented. Here we also show that the dynamical system given by Bender *et al.*[11] is bi-Hamiltonian. In Sec. III we discuss Poisson structures in $\mathbb{R}^n$. We give a representation of the Poisson structure in $\mathbb{R}^n$ in terms of the invariant functions of the vector field $\vec{X}$. Such a representation leads to a classification of dynamical systems with respect to these functions.

II. DYNAMICAL SYSTEMS IN $\mathbb{R}^3$

Although the proof of (2) was given in Ref. 3, here we shall give two simpler proofs. The first one is a shorter proof than the one given in Ref. 3. In the sequel we use the notations $x^1=x$, $x^2=y$, $x^3=z$.

Theorem 1: *All Poisson structures in $\mathbb{R}^3$ have the form (2), i.e., $J^{ij}=\mu\epsilon^{ijk}\partial_k H_0$. Here μ and H_0 are some differentiable functions of x^i and t*, $(i=1,2,3)$.

Proof: Any skew-symmetric second rank tensors in $\mathbb{R}^3$ can be given as

$$J^{ij} = \epsilon^{ijk} J_k, \quad i,j = 1,2,3, \tag{10}$$

where J_1, J_2 and J_3 are differentiable functions in $\mathbb{R}^3$ and we assume that there exists a domain Ω in $\mathbb{R}^3$ so that these functions do not vanish simultaneously. When (10) inserted into the Jacobi equation (1) we get

$$\vec{J} \cdot (\vec{\nabla} \times \vec{J}) = 0, \tag{11}$$

where $\vec{J}=(J_1,J_2,J_3)$ is a differentiable vector field in $\mathbb{R}^3$ not vanishing in Ω. We call $\vec{J}$ as the Poisson vector field. It is easy to show that (11) has a local scale invariance. Let $\vec{J}=\psi\vec{E}$, where ψ is an arbitrary function. If $\vec{E}$ satisfies (11) then $\vec{J}$ satisfies the same equation. Hence it is enough to show that $\vec{E}$ is proportional to the gradient of a function. Using freedom of local scale invariance we can take $\vec{E}=(u,v,1)$, where u and v are arbitrary functions in $\mathbb{R}^3$. Then (11) for vector $\vec{E}$ reduces to

$$\partial_y u - \partial_x v - v\partial_z u + u\partial_z v = 0, \tag{12}$$

where x,y,z are local coordinates. Letting $u=\partial_x f/\rho$ and $v=\partial_y f/\rho$, where f and ρ are functions of x,y,z, we get

$$\partial_x f \partial_y(\rho - \partial_z f) - \partial_y f \partial_x(\rho - \partial_z f) = 0. \tag{13}$$

General solution of this equation is given by

$$\rho - \partial_z f = h(f,z), \tag{14}$$

where h is an arbitrary function of f and z. Then the vector filed $\vec{E}$ takes the form

$$\vec{E} = \frac{1}{\partial_z f + h}(\partial_x f, \partial_y f, \partial_z f + h). \tag{15}$$

Let $g(f,z)$ be a function satisfying $g_{,z}=h\partial_f g$. Here we note that $\partial_z g(f,z)=(\partial g/\partial f)\partial_z f + g_{,z}$, where $g_{,z}=\partial_s g(f(x,y,z),s)|_{s=z}$. Then (15) becomes

$$\vec{E} = \frac{1}{(\partial_z f + h)\partial_f g}\vec{\nabla} g, \tag{16}$$

which completes the proof. Here $\partial_f g = \partial g / \partial f$. ■

The second proof is an indirect one which is given in Ref. 8 (Theorem 5 in this reference).

Definition 2: Let $\vec{F}$ be a vector field in $\mathbb{R}^3$. Then the equation $\vec{F}\cdot d\vec{x}=0$ is called a Pfaffian differential equation. A Pfaffian differential equation is called integrable if the 1-form $\vec{F}\cdot d\vec{x} = \mu dH$, where μ and H are some differentiable functions in $\mathbb{R}^3$.

Let us now consider the Pfaffian differential equation with the Poisson vector field $\vec{J}$ in (10),

$$\vec{J}\cdot d\vec{x} = 0. \tag{17}$$

For such Pfaffian differential equations we have the following result (see Ref. 8).

Theorem 3: *A necessary and sufficient condition that the Pfaffian differential equation $\vec{J}\cdot d\vec{x}=0$ should be integrable is that $\vec{J}\cdot(\vec{\nabla}\times\vec{J})=0$.*

By (11), this theorem implies that $\vec{J}=\mu\vec{\nabla}\Psi$.

A well known example of a dynamical system with Hamiltonian structure of the form (4) is the Euler equations.

Example 1: The Euler equations[6] are

$$\dot{x} = \frac{I_2 - I_3}{I_2 I_3} yz,$$

$$\dot{y} = \frac{I_3 - I_1}{I_3 I_1} xz,$$

$$\dot{z} = \frac{I_1 - I_2}{I_1 I_2} xy, \tag{18}$$

where $I_1, I_2, I_3 \in \mathbb{R}$ are some (nonvanishing) real constants. This system admits Hamiltonian representation of the form (4). The matrix J can be defined in terms of functions $\Psi = H_0 = -\frac{1}{2}(x^2 + y^2 + z^2)$ and $\mu=1$, and we take $H=H_1=x^2/2I_1+y^2/2I_2+z^2/2I_3$.

Writing the Poisson structure in the form (2) allows us to construct bi-Hamiltonian representations of a given Hamiltonian system.

Definition 4: Two Poisson structures J_0 and J_1 are compatible, if the sum J_0+J_1 defines also a Poisson structure.

Lemma 5: Let μ, H_0, and H_1 be arbitrary differentiable functions. Then the Poisson structures J_0 and J_1 given by $J_0^{ij}=\mu\epsilon^{ijk}\partial_k H_0$ and $J_1^{ij}=-\mu\epsilon^{ijk}\partial_k H_1$ are compatible.

This suggests that all Poisson structures in $\mathbb{R}^3$ have compatible companions. Such compatible Poisson structures can be used to construct bi-Hamiltonian systems (for Hamiltonian and bi-Hamiltonian systems see Refs. 6 and 12 and the references therein).

Definition 6: A Hamiltonian equation is said to be bi-Hamiltonian if it admits compatible Poisson structures J_0 and J_1 with the corresponding Hamiltonian functions H_1 and H_0, respectively, such that

$$\frac{dx}{dt} = J_0 \nabla H_1 = J_1 \nabla H_0. \tag{19}$$

Lemma 7: Let J_0 be given by (2), i.e., $J_0^{ij}=\mu\epsilon^{ijk}\partial_k H_0$, and let H_1 be any differentiable function, then the Hamiltonian equation,

$$\frac{dx}{dt} = J_0 \nabla H_1 = J_1 \nabla H_0 = \mu\vec{\nabla} H_1 \times \vec{\nabla} H_0, \tag{20}$$

is bi-Hamiltonian with the second Poisson structure given by J_1 *with entries* $J_1^{ij} = -\mu\epsilon^{ijk}\partial_k H_1$ *and the second Hamiltonian* H_0.

Let us prove that any dynamical system in $\mathbb{R}^3$ has Hamiltonian form.

Theorem 8: *All dynamical systems in* $\mathbb{R}^3$ *are Hamiltonian. This means that any vector field* $\vec{X}$ *in* $\mathbb{R}^3$ *is Hamiltonian vector field. Furthermore, all dynamical systems in* $\mathbb{R}^3$ *are bi-Hamiltonian.*

Proof: Let ζ be an invariant function of the vector field $\vec{X}$, i.e., $X(\zeta) \equiv \vec{X} \cdot \vec{\nabla}\zeta = 0$. This gives a first order linear differential equation in $\mathbb{R}^3$ for ζ. For a given vector field $\vec{X} = (f, g, h)$ this equation becomes

$$f(x,y,z,t)\frac{\partial \zeta}{\partial x} + g(x,y,z,t)\frac{\partial \zeta}{\partial y} + h(x,y,z,t)\frac{\partial \zeta}{\partial z} = 0, \tag{21}$$

where x, y, z are local coordinates. From the theory of first order linear partial differential equations,[6–8] the general solution of this partial differential equation can be determined from the following set of equations:

$$\frac{dx}{f(x,y,z,t)} = \frac{dy}{g(x,y,z,t)} = \frac{dz}{h(x,y,z,t)}. \tag{22}$$

There exist two functionally independent solutions ζ_1 and ζ_2 of (22) in an open domain $D \subset \mathbb{R}^3$ and the general solution of (21) will be an arbitrary function of ζ_1 and ζ_2, i.e., $\zeta = F(\zeta_1, \zeta_2)$. This implies that the vector field $\vec{X}$ will be orthogonal to both $\vec{\nabla}\zeta_1$ and $\vec{\nabla}\zeta_2$. Then $\vec{X} = \mu(\vec{\nabla}\zeta_1) \times (\vec{\nabla}\zeta_2)$. Hence the vector field $\vec{X}$ is Hamiltonian by (5). ■

This theorem gives also an algorithm to find the Poisson structures or the functions H_0, H_1, and μ of a given dynamical system. The functions H_0 and H_1 are the invariant functions of the vector field $\vec{X}$ which can be determined by solving the system equations (22) and μ is determined from

$$\mu = \frac{\vec{X} \cdot \vec{X}}{\vec{X} \cdot (\vec{\nabla}H_0 \times \vec{\nabla}H_1)}. \tag{23}$$

Note that μ can also be determined from

$$\mu = \frac{X^1}{\partial_2 H_0 \partial_3 H_1 - \partial_3 H_0 \partial_2 H_1} = \frac{X^2}{\partial_3 H_1 \partial_3 H_1 - \partial_1 H_0 \partial_3 H_1} = \frac{X^3}{\partial_1 H_0 \partial_2 H_1 - \partial_2 H_0 \partial_1 H_1}. \tag{24}$$

Example 2: As an application of the method described above we consider Kermac–Mckendric system,

$$\dot{x} = -rxy,$$

$$\dot{y} = rxy - ay,$$

$$\dot{z} = ay, \tag{25}$$

where $r, a \in \mathbb{R}$ are constants. Let us put the system into Hamiltonian form. For the Kermac–Mckendric system, Eqs. (22) become

$$\frac{dx}{-rxy} = \frac{dy}{rxy - ay} = \frac{dz}{ay}. \tag{26}$$

Here a and r may depend on t, in general. Adding the numerators and denominators of (26) we get

$$\frac{dx}{-rxy} = \frac{dx+dy+dz}{0}. \tag{27}$$

Hence $H_1=x+y+z$ is one of the invariant functions of the vector field. Using the first and last terms in (26) we get

$$\frac{dx}{-rx} = \frac{dz}{a}, \tag{28}$$

which gives $H_0=rz+a\ln x$ as the second invariant function of the vector field $\vec{X}$. Using (23) we get $\mu=xy$. Since $\vec{X}=\mu\vec{\nabla}H_0\times\vec{\nabla}H_1$, the system admits a Hamiltonian representation where the Poisson structure J is given by (2) with $\mu=xy$, $\Psi=H_0=rz+a\ln x$, and the Hamiltonian is $H_1=x+y+z$.

Example 3: The dynamical system is given by

$$\dot{x} = yz(1+2x^2N/D),$$

$$\dot{y} = -2xz(1-y^2N/D),$$

$$\dot{z} = xy(1+2z^2N/D), \tag{29}$$

where $N=x^2+y^2+z^2-1$, $D=x^2y^2+y^2z^2+4x^2z^2$. This example was obtained by Bender *et al.*[11] by complexifying the Euler system in Example 1. They claim that this system is not Hamiltonian apparently bearing in mind the more classical definition of a Hamiltonian system. Using the Definition 6 we show that this system is not only Hamiltonian but also bi-Hamiltonian. We obtain that

$$H_0 = \frac{(N+1)^2}{D}N, \quad H_1 = \frac{x^2-z^2}{D}(2y^2z^2+4x^2z^2+y^4+2x^2y^2-y^2). \tag{30}$$

Here

$$\mu = \frac{D^2}{4[3D^2+DP+Q]}, \tag{31}$$

where

$$P = -2x^4+4y^4-4x^2y^2+x^2-2y^2-4y^2z^2+14z^4+z^2,$$

$$Q = -2x^8+12x^6z^2+2x^6-20x^4z^4-6x^4z^2-52x^2z^6-6x^2z^4+y^8-y^6+4y^4z^4-16y^2z^6-2z^8+2z^6. \tag{32}$$

Indeed these invariant functions were given in Ref. 11 as functions A and B. The reason why Bender *et al.*[11] concluded that the system in Example 3 is non-Hamiltonian is that the vector field $\vec{X}$ has nonzero divergence. It follows from $\vec{X}=\mu\vec{\nabla}H_0\times\vec{\nabla}H_1$ that $\vec{\nabla}\cdot((1/\mu)\vec{X})=0$. When μ is not a constant the corresponding Hamiltonian vector field has a nonzero divergence.

Remark 1: With respect to the time dependence of invariant functions of the vector field $\vec{X}$ dynamical systems in $\mathbb{R}^3$ can be split into three classes.

Class A: Both invariant functions H_0 and H_1 of the vector field $\vec{X}$ do not depend on time explicitly. In this case both H_0 and H_1 are also invariant functions of the dynamical systems. Hence the system is superintegrable. All autonomous dynamical systems such as the Euler equation (Example 1) and the Kermac–Mckendric system (Example 2) belong to this class.

Class B: One of the invariant functions H_0 and H_1 of the vector field $\vec{X}$ depends on t explicitly. Hence the other one is an invariant function also of the dynamical system. When I_1, I_2 and I_3 in

Example 1 are time dependent the Euler system becomes the member of this class. In this case H_0 is the Hamiltonian function and H_1 is the function defining the Poisson structure. Similarly, in Example 2 we may consider the parameters a and r as time dependent. Then Kermac–Mckendric system becomes also a member of this class.

Class C: Both H_0 and H_1 are explicit functions of time variable t but they are not the invariants of the system. There may be invariants of the dynamical system. Let F be such an invariant. Then

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + \{F,H_1\}_0 = \frac{\partial F}{\partial t} + \{F,H_0\}_1 = 0, \tag{33}$$

where for any F and G,

$$\{F,G\}_\alpha \equiv J^{ij}_\alpha \partial_i F \partial_j G, \quad \alpha = 0,1. \tag{34}$$

III. POISSON STRUCTURES IN $\mathbb{R}^n$

Let us consider the dynamical system

$$\frac{dx^i}{dt} = X^i(x^1,x^2, \ldots ,x^n,t), \quad i = 1,2, \ldots ,n. \tag{35}$$

Theorem 9: *All dynamical systems in $\mathbb{R}^n$ are Hamiltonian. Furthermore, all dynamical systems in $\mathbb{R}^n$ are $(n-1)$ -Hamiltonian.*

Proof: Extending the proof of Theorem 8 to $\mathbb{R}^n$ consider the linear partial differential equation (8). There exist $n-1$ functionally independent solutions H_α, $(\alpha=1,2,\ldots,n-1)$ of this equation (which are invariant functions of the vector field $\vec{X}$).[6–8] Since $\vec{X}$ is orthogonal to the vectors $\vec{\nabla} H_\alpha$ $(\alpha=1,2,\ldots,n-1)$, we have

$$\vec{X} = \mu \begin{bmatrix} \vec{e_1} & \vec{e_2} & \cdot & \cdot & \cdot & \vec{e_n} \\ \partial_1 H_1 & \partial_2 H_1 & \cdot & \cdot & \cdot & \partial_n H_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \partial_1 H_{n-1} & \partial_2 H_{n-1} & \cdot & \cdot & \cdot & \partial_n H_{n-1} \end{bmatrix}, \tag{36}$$

where the function μ is a coefficient of proportionality and $\vec{e}_i$ is n-dimensional unit vector with the ith coordinate 1 and remaining coordinates 0. Therefore,

$$X^i = \mu \epsilon^{ij_1 j_2 \cdots j_{n-1}} \partial_{j_1} H_1 \partial_{j_2} H_2 \cdots \partial_{j_{n-1}} H_{n-1}. \tag{37}$$

Hence all dynamical systems (35) have the Hamiltonian representation

$$\frac{dx^i}{dt} = J^{ij}_\alpha \partial_j H_\alpha, \quad i = 1,2, \ldots ,n \quad (\text{no sum on } \alpha) \tag{38}$$

with

$$J^{ij}_\alpha = \mu \epsilon^{\alpha\alpha_1\alpha_2\cdots\alpha_{n-2}} \epsilon^{ijj_1\cdots j_{n-2}} \partial_{j_1} H_{\alpha_1} \partial_{j_2} H_{\alpha_2} \cdots \partial_{j_{n-2}} H_{\alpha_{n-2}}, \tag{39}$$

where $i,j=1,2,\ldots,n$, $\alpha=1,2,\ldots,n-1$. Here $\epsilon^{ijj_1\cdots j_{n-2}}$ and $\epsilon^{\alpha\alpha_1\alpha_2\cdots\alpha_{n-2}}$ are Levi–Civita symbols in n and $n-1$ dimensions, respectively. The function μ can be determined, for example, from

$$\mu = \frac{X^1}{\begin{vmatrix} \partial_2 H_1 & \cdot & \cdot & \cdot & \partial_n H_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \partial_2 H_{n-1} & \cdot & \cdot & \cdot & \partial_n H_{n-1} \end{vmatrix}}. \tag{40}$$

It can be seen that the matrix J_α with the entries J_α^{ij} given by (39) defines a Poisson structure in $\mathbb{R}^n$ and since

$$J_\alpha \cdot \nabla H_\beta = 0, \quad \alpha, \beta = 1, 2, \dots, n-1, \tag{41}$$

with $\beta \neq \alpha$, the rank of the matrix J_α equals 2 (for all $\alpha = 1, 2, \dots, n-1$). In (38) we can take any of $H_1, H_2, \dots, H_{n-1}$ as the Hamilton function and use the remaining H_k's in (39). We observe that all dynamical systems (35) in $\mathbb{R}^n$ have $n-1$ number of different Poisson structures in the form given by (39). The same system may have a Poisson structure with a rank higher than 2. The following example clarifies this point.

Example 4: Let

$$\dot{x}^1 = x^4, \quad \dot{x}^2 = x^3, \quad \dot{x}^3 = -x^2, \quad \dot{x}^4 = -x^1. \tag{42}$$

Clearly this system admits a Poisson structure with rank 4,

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad H = \frac{1}{2}[(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2]. \tag{43}$$

The invariant functions of the vector field $\vec{X} = (x^4, x^3, -x^2, -x^1)$ are

$$H_1 = \tfrac{1}{2}[(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2], \tag{44}$$

$$H_2 = \tfrac{1}{2}[(x^2)^2 + (x^3)^2), \tag{45}$$

$$H_3 = x^1 x^3 - x^2 x^4. \tag{46}$$

Then the above system has three different ways of representation with the second rank Poisson structures,

$$J_1^{ij} = \mu \epsilon^{ijkl} \partial_k H_1 \partial_l H_2, \quad H = H_3, \tag{47}$$

$$J_2^{ij} = -\mu \epsilon^{ijkl} \partial_k H_1 \partial_l H_3, \quad H = H_2, \tag{48}$$

$$J_3^{ij} = \mu \epsilon^{ijkl} \partial_k H_2 \partial_l H_3, \quad H = H_1, \tag{49}$$

where $\mu(x^1 x^2 + x^3 x^4) = 1$. These Poisson structures are compatible not only pairwise but also triple-wise. This means that any linear combination of these structures is also a Poisson structure. Let $J = \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3$ then it is possible to show that

$$J^{ij} = \mu \epsilon^{ijkl} \partial_k \tilde{H}_1 \partial_l \tilde{H}_2, \tag{50}$$

where $\tilde{H}_1$ and $\tilde{H}_2$ are linear combinations of H_1, H_2, and H_3,

$$\tilde{H}_1 = H_1 - \frac{\alpha_3}{\alpha_2} H_2, \quad \tilde{H}_2 = \alpha_1 H_2 - \alpha_2 H_3 \quad \text{if} \quad \alpha_2 \neq 0, \tag{51}$$

$$\tilde{H}_1 = \alpha_1 H_1 - \alpha_3 H_2, \quad \tilde{H}_2 = H_2 \quad \text{if} \quad \alpha_2 = 0. \tag{52}$$

Definition 10: A dynamical system (35) in $\mathbb{R}^n$ is called superintegrable if it has $n-1$ functionally independent first integrals (constants of motion).

Theorem 11: *All autonomous dynamical systems in $\mathbb{R}^n$ are superintegrable.*

Proof: If the system (35) is autonomous, then the vector field $\vec{X}$ does not depend on t explicitly. Therefore, each of the invariant functions H_α $(\alpha=1,2,\ldots,n-1)$ of the vector field $\vec{X}$ is a constant of motion of the system (35).

Some (or all) of the invariant functions H_α $(\alpha=1,2,\ldots,n-1)$ of the vector field $\vec{X}$ may depend on t. Like in $\mathbb{R}^3$ we can classify the dynamical systems in $\mathbb{R}^n$ with respect to the invariant functions of the vector field $\vec{X}(x^1,x^2,\ldots,x^n,t)$.

Class A: All invariant functions H_α $(\alpha=1,2,\ldots,n-1)$ of the vector field $\vec{X}$ do not depend on t explicitly. In this case all functions H_α $(\alpha=1,2,\ldots,n-1)$ are also invariant functions (constants of motion) of the dynamical system. Hence the system is superintegrable. In the context of the multi-Hamiltonian structure, such systems were first studied by Refs. 9 and 10. The form (39) of the Poisson structure was given in these works. Its properties were investigated in Ref. 13.

Class B: At least one of the invariant functions H_α $(\alpha=1,2,\ldots,n-1)$ of the vector field $\vec{X}$ does not depend on t explicitly. That function is an invariant function also of the dynamical system.

Class C: All H_α $(\alpha=1,2,\ldots,n-1)$ are explicit functions of time variable t but they are not the invariants of the system. There may be invariants of the dynamical system. Let F be such an invariant. Then

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + \{F,H_\alpha\}_\alpha = 0, \quad \alpha = 1,2,\ldots,n-1, \tag{53}$$

where for any F and G

$$\{F,G\}_\alpha \equiv J_\alpha^{ij} \partial_i F \partial_j G, \quad \alpha = 0,1,\ldots,n-1. \tag{54}$$

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