MATH. 443
PARTIAL DIFFERENTIAL EQUATIONS

Partial Differential Equations with Constant Coefficients: Ref. Sneddon: Introduction to Partial Differential Equations .

Let the independent variables be x and y and the dependent variable be z. Let

$$\frac{\partial}{\partial x} = D, \quad \frac{\partial}{\partial y} = D' \tag{1}$$

A linear partial differential differential equation is given by

$$F(D, D')\, z = f(x, y) \tag{2}$$

where $F(u, v)$ is a polynomial of u and v. Thus means that $F(D, D')$ is polynomial of the differential operators D and D'. Hence it is an operator acting on z.

Definition 1. *A differential operator is called reducible if it can be written as a product of linear factors $L_i = \alpha_i\, D + \beta_i\, D' + \gamma_i$, $i = 1, 2, \cdots, N$, where $N =$ is the order of the F operator and α_i, β_i and γ_i (for all $i = 1, 2, \cdots, N$) are constants. If an operator is not reducible it is called irreducible.*

Hence a reducible operator $F(D, D')$ can be written as

$$F(D, D') = L_1^{m_1}\, L_2^{m_2}\, \cdots\, L_n^{m_n} = \Pi_{r=1}^{n}\, L_r^{m_r} \tag{3}$$

where $m_1 + m_2 + \cdots + m_n = N$. Hence m_r represents the multiplicity that the linear factor L_r occurs in the operator F.

Example 1. Examples of reducible operators

$$F(D, D') = D^2 - D'^{\,2} = (D - D')(D + D'), \tag{4}$$
$$F(D, D') = D^2 - D\, D' = 2D'^{\,2} D + D' = (D - 2D' + 1)(D + D'), \tag{5}$$
$$F(D, D') = D^4 - D^2\, D'^{\,2} + D'^{\,2} = (D - D')^2\, (D + D')^2 \tag{6}$$

Example 2. Examples of irreducible operators

$$F(D, D') = D^2 - D', \tag{7}$$
$$F(D, D') = D^4 + D'^{\,4}, \tag{8}$$
$$F(D, D') = D^4 + D'^{\,2} \tag{9}$$

Theorem 2. *The linear factors with constant coefficients commute. That is, if*

$$L_i = \alpha_i\, D + \beta_i\, D' + \gamma_i, \quad i = 1, 2, \cdots, N, \tag{10}$$

where α_i, β_i and γ_i (for all $i = 1, 2, \cdots, N$) are constants, then

$$L_i\, L_j = L_j\, L_i, \quad i \neq j \tag{11}$$

Hence the order in the product expression (3) is not so important. We can change the the places of linear factors L_i's in (3) freely.

This theorem is very important. It enables us to find the general solution of the homogenous equation

$$F(D, D')\, z = 0 \tag{12}$$

Theorem 3. *The general solution of (12) is given as*

$$z(x, y) = z_1(x, y) + z_2(x, y) + \cdots + z_n(x, y) \tag{13}$$

where $z_r(x, y),\ r = 1, 2, \cdots n$ are the solutions of the equations

$$L_r^{m_r}(z_r) = 0, \quad r = 1, 2, \cdots, n \tag{14}$$

Hence it is clear that we have to know how to solve such equations. The Lagrange method is the most effective one to find these solutions. Let us see some examples.

Example 3. Solve

$$(D + 2D' - 3)(D + D')\, z = 0 \tag{15}$$

Solution: $L_1 = D + 2D' - 3$, $L_2 = D + D'$. Let z_1 and z_2 solve the equations

$$L_1(z_1) = z_{1,x} + 2z_{1,y} - 3z_1 = 0, \tag{16}$$

$$L_2(z_2) = z_{2,x} + z_{2,y} = 0 \tag{17}$$

These are linear first order partial differential equations which can be easily solved by the use of Lagrange method.

$$
\begin{aligned}
z_1(x,y) &= e^{3x}\,\phi_1(2x-y), && (18)\\
z_2(x,y) &= \phi_2(x-y) && (19)
\end{aligned}
$$

Hence the solution of the problem

$$z(x,y) = e^{3x}\,\phi_1(2x-y) + \phi_2(x-y) \tag{20}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Example 4. Solve

$$(D+2D'-3)(D+D')^2\,z = 0 \tag{21}$$

Solution: $L_1 = D+2D'-3$, $L_2 = D+D'$, $m_2 = 2$. Let z_1 and z_2 solve the equations

$$
\begin{aligned}
L_1(z_1) &= z_{1,x} + 2z_{1,y} - 3z_1 = 0, && (22)\\
L_2^2(z_2) &= (D+D')^2\,z_2 = 0 && (23)
\end{aligned}
$$

We know the solution of the equation (30). Let us solve the equation (36). Let $w = (D+D')z_2$ then

$$(D+D')w = 0 \tag{24}$$

which has the solution

$$w(x,y) = \phi_1(x-y) \tag{25}$$

Hence z_2 satisfies

$$(D+D')z_2 = \phi_2(x-y) \tag{26}$$

with the solution

$$z_2(x,y) = \phi_2(x-y)\,x + \phi_3(x-y) \tag{27}$$

Then the solution of the problem is

$$z(x,y) = e^{3x}\,\phi_1(2x-y) + \phi_2(x-y)\,x + \phi_3(x-y) \tag{28}$$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Example 5. Solve

$$(D + 2D' - 3)^3 (D + D')^4 z = 0 \tag{29}$$

Solution: $L_1 = D + 2D' - 3$, $L_2 = D + D'$, $m_1 = 3$ and $m_2 = 4$. Let z_1 and z_2 solve the equations

$$\begin{aligned} L_1^3(z_1) &= (D + 2D' - 3)^3 z_1 = 0, & (30)\\ L_2^4(z_2) &= (D + D')^4 z_2 = 0 & (31) \end{aligned}$$

Hence

$$\begin{aligned} z_1(x, y) &= e^{3x} [\phi_1(2x - y) x^2 + \phi_2(2x - y) + \phi_3(2x - y)] & (32)\\ z_2(x, y) &= \phi_4(x - y) x^3 + \phi_5(x - y) x^2 + +\phi_6(x - y) x + \phi_7(x - y) \end{aligned}$$

Then the solution is —

$$\begin{aligned} z(x, y) &= e^{3x} [\phi_1(2x - y) x^2 + \phi_2(2x - y) + \phi_3(2x - y)] & (33)\\ &+\phi_4(x - y) x^3 + \phi_5(x - y) x^2 + \phi_6(x - y) x & (34)\\ &+\phi_7(x - y) & (35) \end{aligned}$$

where $\phi_1, \phi_2, \cdots, \phi_7$ are arbitrary functions.

When the partial differential equation is not homogenous we have the following:

Theorem 4. *Let*

$$F(D, D') z = f(x, y) \tag{36}$$

where f is a given function of x and y and F is a reducible operator. The solution of this partial differential equation is given by

$$z(x, y) = z_h(x, y) + z_p(x, y) \tag{37}$$

where z_h is the solution of the homogenous equation and z_p, the particular solution, is a solution of the inhomogeneous equation (36)

We know how to find, z_h, the solution of the homogenous equation. To find a particular solutio, z_p of (36) we have the following methods.

a) By inspection: For some simple function $f(x)$, in particular when it is a monomial of x and y or exponential function of x and y we can assume a $z_p(x, y)$

Example 6. Solve

$$(D + 2D')(D + D')\, z = 5\, x^2 \tag{38}$$

Solution: One can guess that $z_p(x, y) = a\, x^4$. Inserting it into the differential equation we get $a = \frac{5}{12}$. Hence the solution is

$$z(x, y) = \frac{5}{12}\, x^4 + \phi_1(2x - y) + \phi_2(x - y) \tag{39}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

c) By Lagrange Method: The Lagrange method can be used for all $f(x, y)$, but it is quite lengthy if the order of F operator is larger then two.

Example 7. Solve

$$(D + 2D')(D + D')\, z = x + 2y \tag{40}$$

Solution: To find a particular solution z_p, first let $w = (D + D')\, z_p$. Then

$$(D + 2D')\, w = x + 2y \tag{41}$$

We can now easily solve w by applying the Lagrange method. While using the Lagrange method we can ignore all arbitrary functions. The purpose is to get a particular solution. For instance the solution of the above equation is

$$w = -\frac{3}{2}\, x^2 + 2xy + \phi(2x - y) \tag{42}$$

where ϕ is an arbitrary function. Here we can ignore ϕ and let $w = -\frac{3}{2}\, x^2 + 2xy$. Then z_p satisfies

$$(D + D')\, z_p = -\frac{3}{2}\, x^2 + 2xy \tag{43}$$

Using the Lagrange method once more, a solution of this equations yields

$$z_p = -\frac{5}{6}\, x^3 + x^2\, y \tag{44}$$

Then the full solution is

$$z(x,y) = z_h + z_p = -\frac{5}{6}\,x^3 + x^2\,y + \phi_1(2x-y) + \phi_2(x-y) \tag{45}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

c) By Inverting F: If the function f is a polynomial of x and y we write the solution of $F(D,D')\,z_p = f(x,y)$ as

$$z_p(x,y) = F^{-1}(D,D')f \tag{46}$$

Inversion of F is not so easy when its order is greater then two.

Example 8. Solve

$$(D+2D')(D+D')\,z = x+2y \tag{47}$$

Solution: This is the same example as Example 7. To find z_p we shall use the inversion of F. We do it in the following way

$$F = (D+2D')(D+D') = D^{-2}(1+2\,X)(1+X) \tag{48}$$

where $D\,D^{-1} = \textit{identity}$, hence D^{-1} is the integration with respect to x variable and $X = D^{-1}\,D'$. Then

$$\begin{aligned} F^{-1}\,f &= D^{-2}\,(1+2\,X)^{-1}\,(1+X)^{-1}\,f, && (49)\\ & \quad D^{-2}\,(1+2X-4X^2+\cdots)(1-X+X^2-\cdots)\,f && (50) \end{aligned}$$

Since $X(f) = 2x$ and $X^n(f) = 0$ for all $n \geq 2$. Then we have

$$\begin{aligned} z_p &= F^{-1}\,f = D^{-2}\,(1-2X)(f-X(f)), && (51)\\ &= D^{-2}(1-2X)(-x+2y) = D^{-2}(-5x+2y) = -\frac{5}{6}\,x^3 + x^2\,y && (52) \end{aligned}$$

which is the same solution we obtained in Example 7 by the use of Lagrange method.