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Cartan ideal, prolongation, and Bäcklund transformations for Einstein's equations

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Einstein's equations in the Newman–Penrose formalism for vacuum, vacuum with cosmological constant, and electrovacuum fields are expressed as Cartan ideals. Two different prolongations of these ideals are obtained. These two types of prolonged ideals generalize previous prolongations for vacuum fields to vacuum with cosmological constant and electrovacuum fields. Some Bäcklund transformations are obtained for vacuum, vacuum with cosmological constant, and electrovacuum fields. These Bäcklund transformations include the generalized Kerr–Schild (GKS) transformation, and a two-parameter generalization of the GKS transformation. GKS transformations are studied in detail. Expressions for the transformation of Newman–Penrose quantities are given and algebraic properties are discussed. It is shown that the GKS transformation cannot give algebraically general and asymptotically flat vacuum and electrovacuum space-time metrics.

I. INTRODUCTION

The geometric theory of partial differential equations (PDE's) as found by Cartan,^{1–3} besides its own interest, is a useful tool for inverse scattering problems and construction of Bäcklund transformations. The essence of Cartan's approach is to express PDE's as a differential ideal.⁴ Then, prolongating this ideal,^{5–7} it is possible to obtain "associated equations of the PDE," which are used in solving the original PDE's via inverse scattering methods, or in establishing a correspondence between solutions of the PDE's (i.e., Bäcklund correspondence).

Inverse scattering technique⁸ is a well-understood and powerful tool in two-dimensional problems. Extension of this method to higher dimensions is also subject to current interest.⁹ The first step in the application of the inverse scattering technique is to find the associated linear equation for the PDE's, i.e., the linear equation whose integrability is guaranteed by the original PDE's. For problems in higher dimensions existence of associated linear equations is not sufficient to solve the PDE's, but they may lead to Bäcklund transformations. The prolongation technique mentioned above can be used to obtain these associated equations. Prolongation was first described by Cartan as a lifting of the ideal representing the PDE's to a fiber bundle, and used to eliminate independent variables. A nontrivial generalization is given by Estabrook and Wahlquist¹⁰ as prolongating the ideal (representing the PDE's) by lifting and adding new generators. These additional generators represent associated equations for the original PDE's.

Applications of inverse scattering to general relativity is practically restricted to space-times admitting symmetries.

In fact, space-times admitting two commuting non-null Killing vector fields has been shown to be completely integrable^{11,12} and their associated linear equations led to several equivalent Bäcklund transformations.^{13–15} The applications of prolongation technique to general relativity starts with Harrison, by the construction of Bäcklund transformations for the Ernst equation.¹³ Later work, from China¹⁶ and Gürses,¹⁷ generalize this construction to space-times without symmetries satisfying vacuum Einstein equations: China proposes a linear equation to be useful in inverse scattering (see also Julia¹⁸). On the other hand, Gürses' approach leads to the construction of Bäcklund transformations,^{17,19} which are generalized to Einstein spaces (vacuum with cosmological constant) and electrovacuum fields in the present work.

Prolongation technique, as given by China, Gürses, and Harrison^{16,17,20,21} make use of a compact matrix formulation of Einstein's equations obtained from connections in principal bundles. In fact, starting from a null basis for tetrad vectors, the expressions obtained constitute a compact version of the Newman–Penrose (NP) formalism.²² Expression of Einstein's equations for various (vacuum, vacuum with cosmological constant, electrovacuum) cases in compact form are given in Sec. II.

The formalism described above is used to study Einstein's equations in the framework of Cartan's theory of differential equations: The PDE's in terms of differential forms obtained in Sec. II are used to construct a closed ideal in Sec. III. Then, in Sec. IV prolonged ideals are investigated, and two different prolongations of the ideals constructed in Sec. III are given. One of the prolongations generalizes the equations obtained by China¹⁶ to nonvacuum fields,¹⁹ and the other gives Bäcklund transformations that are discussed in detail in Sec. V. In Sec. V, some Bäcklund transformations are constructed. Construction of the Bäcklund transformation is not sufficient to ensure the existence of new solutions

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since it still remains to solve the associated equations that may well lead (since the system to be solved may be overdetermined) to nontrivial restrictions of the background (i.e., the known solution to start with) or to a trivial solution for the transformation itself. This last step, that is, the solution of associated equations, is presented in detail for the GKS transformation in Sec. VI. The compatibility condition of the GKS transformation for algebraic general backgrounds is found and examples for space-times satisfying these conditions are given. The main feature of this constraint is that vacuum (and in general electrovacuum) backgrounds admitting the GKS transformation cannot be both algebraic general and asymptotically flat,^{22,25,26} and since the constraint is invariant under the GKS transformation resulting space-times also will have the same character.²⁴

II. EINSTEIN'S EQUATIONS IN THE COMPACT NEWMAN-PENROSE FORMALISM

In this section we will give a compact formulation for the structure equations of a space-time and the Einstein equations. We will essentially express Newman-Penrose (NP) equations²² in terms of $\mathfrak{sl}(2, \mathbb{C})$ -valued forms.²⁷ Then Einstein's equations are obtained simply by equating components of the tracefree Ricci spinor and curvature scalar to corresponding spinorial components of the energy momentum tensor. The crucial point of the formulation is that we have to write these equalities as matrix-valued differential forms that will be suitable for the expression of equations of motion for physical (nongravitational) fields.¹⁷

In the following we denote by M a space-time manifold with a Lorentz metric g . The bundle of orthonormal frames on M is a principal fiber bundle^{28,29(a)} with structure group $\text{SO}(1,3)$. Then the connection form of M takes its values in the Lie algebra of $\text{SO}(1,3)$, and in terms of the canonical form of the frame bundle, structure equations of M can be written in a compact form as

$$de = -\omega \wedge e, \tag{2.1a}$$

$$d\omega = -\omega \wedge \omega + \Omega. \tag{2.1b}$$

In the equations above, e is an \mathbb{R}^4 valued one-form. In fact the orthonormal tetrad one-form $e = \{e_k\}$, where $e_k = e_{k\mu} dx^\mu$ in a coordinate basis with the metric given by

$$g = e_0 \otimes e_0 - e_1 \otimes e_1 - e_2 \otimes e_2 - e_3 \otimes e_3, \tag{2.2}$$

where \otimes denotes tensor product, and ω and Ω are, respectively, connection and curvature forms taking values in the Lie algebra of $\text{SO}(1,3)$. In Eq. (2.1) d is the exterior derivative and \wedge denotes exterior multiplication of corresponding matrices. In the following we will drop the \wedge sign for exterior products, and all multiplications should be understood as appropriate matrix multiplications with exterior products of corresponding forms, unless otherwise stated.

To obtain a compact NP formalism, we will first establish correspondence between an orthonormal frame $\{e_k\}$ and a 2×2 Hermitian matrix whose entries constitute a null frame, then we will give structure equations in terms of this matrix and $\mathfrak{sl}(2, \mathbb{C})$ -valued forms, and finally we will present transformations of the connection and curvature resulting from the action of $\text{SL}(2, \mathbb{C})$ on null frames.

We establish a 1-1 correspondence between the orthonormal frame $\{e_k\}$ and 2×2 Hermitian matrix σ as

$$\sigma = \frac{1}{\sqrt{2}} \sum_{k=0}^3 (\sigma_k e_k), \tag{2.3a}$$

$$e_k = (1/\sqrt{2}) \text{Tr}(\sigma \sigma_k), \quad k = 0, \dots, 3, \tag{2.3b}$$

where the σ_k are Pauli spin matrices, i.e.,

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.4}$$

We then define a null frame $\{l, n, m, \bar{m}\}$ as

$$\begin{aligned} l &= \frac{(e_0 + e_3)}{\sqrt{2}}, & n &= \frac{(e_0 - e_3)}{\sqrt{2}}, \\ m &= \frac{(e_1 + ie_2)}{\sqrt{2}}, & \bar{m} &= \frac{(e_1 - ie_2)}{\sqrt{2}}, \end{aligned} \tag{2.5}$$

with the metric given by

$$g = l \otimes n + n \otimes l - m \otimes \bar{m} - \bar{m} \otimes m. \tag{2.6}$$

Then Eqs. (2.3a) and (2.5) give an identification of the null frame $\{l, n, m, \bar{m}\}$ with 2×2 Hermitian matrices as

$$\sigma = \begin{pmatrix} l & m \\ \bar{m} & n \end{pmatrix}. \tag{2.7}$$

It can be seen that any other identification of the null frame $\{l, n, m, \bar{m}\}$ with Hermitian 2×2 matrices is similar, either to σ or its complex conjugate $\bar{\sigma}$. We will write all equations in terms of $\bar{\sigma}$ and $\bar{\sigma} = \epsilon \sigma \epsilon^\dagger$ (see Ref. 30), where $\epsilon = -i\sigma_2$. That is,

$$\bar{\sigma} = \begin{pmatrix} n & -\bar{m} \\ -m & l \end{pmatrix}, \quad \bar{\sigma} = \begin{pmatrix} l & \bar{m} \\ m & n \end{pmatrix}. \tag{2.8a}$$

In the following, $\bar{\eta}$ and η^\dagger denote complex and Hermitian conjugates of η respectively, and $\bar{\eta}$ is equal to $\epsilon \eta \epsilon^\dagger$, for any η . Also we have

$$\bar{\Gamma} = -\epsilon \Gamma^\dagger \epsilon^\dagger = -\bar{\Gamma}^\dagger, \tag{2.8b}$$

$$\bar{\phi} = -\epsilon \phi^\dagger \epsilon^\dagger = -\bar{\phi}^\dagger. \tag{2.8c}$$

We now give structure equations in terms of $\bar{\sigma}$ and $\mathfrak{sl}(2, \mathbb{C})$ -valued connection and curvature forms Γ and R , as

$$d\bar{\sigma} + \Gamma \bar{\sigma} - \bar{\sigma} \Gamma^\dagger = 0, \tag{2.9a}$$

$$d\Gamma + \Gamma \Gamma - R = 0, \tag{2.9b}$$

and we obtain Bianchi identities by taking exterior derivative of Eqs. (2.9a) and (2.9b) as²⁷

$$R \bar{\sigma} + \bar{\sigma} R^\dagger = 0, \tag{2.10a}$$

$$dR - R \Gamma + \Gamma R = 0, \tag{2.10b}$$

with

$$\Gamma = \begin{pmatrix} \Gamma_0 & \Gamma_2 \\ \Gamma_1 & -\Gamma_0 \end{pmatrix}, \quad R = \begin{pmatrix} R_0 & R_2 \\ R_1 & -R_0 \end{pmatrix}, \tag{2.11}$$

where

$$\Gamma_0 = \gamma l + \epsilon n - \alpha m - \beta \bar{m}, \tag{2.12a}$$

$$\Gamma_1 = -\tau l - \kappa n + \rho m + \sigma \bar{m}, \tag{2.12b}$$

$$\Gamma_2 = \nu l + \pi n - \lambda m - \mu \bar{m}; \tag{2.12c}$$

$$R_0 = (\Lambda - \phi_{11} - \psi_2) l n + \psi_3 l m + \phi_{12} l \bar{m}$$

$$-\phi_{10}nm - \psi_1 n\bar{m} + (\psi_2 - \phi_{11} - \Lambda)m\bar{m}, \quad (2.13a)$$

$$R_1 = (\psi_1 + \phi_{01})ln - (\psi_2 + 2\Lambda)lm - \phi_{02}l\bar{m} + \phi_{00}nm + \psi_0 n\bar{m} + (\phi_{01} - \psi_1)m\bar{m}, \quad (2.13b)$$

$$R_2 = -(\psi_3 + \phi_{21})ln + \psi_4 lm + \phi_{22}l\bar{m} - \phi_{20}nm - (\psi_2 + 2\Lambda)n\bar{m} - (\phi_{21} - \psi_3)m\bar{m}. \quad (2.13c)$$

In Eqs. (2.12), the scalars $\alpha, \beta, \gamma, \lambda, \mu, \nu, \rho, \sigma, \epsilon, \tau, \kappa, \pi$ are NP spin coefficients, and in Eqs. (2.13), ψ_A ($A = 0, \dots, 4$), ϕ_{ij} ($i, j = 0, \dots, 2$) are, respectively, components of the Weyl and tracefree Ricci spinor and Λ is the curvature scalar.

We then give transformations of Γ and R corresponding to the $SL(2, C)$ action on null frames. If $S \in SL(2, C)$, then the action of S on null frames is given as

$$\tilde{\sigma} \rightarrow S\tilde{\sigma}S^\dagger. \quad (2.14)$$

This action leaves the metric invariant. The connection Γ and the curvature R transform as

$$\Gamma \rightarrow STS^{-1} - dSS^{-1}, \quad (2.15a)$$

$$R \rightarrow SRS^{-1}. \quad (2.15b)$$

Explicit expressions for Eqs. (2.10), (2.15a), and (2.15b) can be found, for example, in Ref. 31.

Finally we will express Einstein's equations in the NP formalism. They are

$$\phi_{ij} = \tau_{ij}, \quad \Lambda = \tau_\Lambda, \quad (2.16)$$

where τ_{ij} and τ_Λ are the corresponding spinorial components of the energy momentum tensor. Thus to obtain a compact formulation we have to separate tracefree Ricci and curvature scalar components of R by a "compact operation," which is exactly multiplication from the right by $\tilde{\sigma}$. We remark that [Eq. (2.10a)] $R\tilde{\sigma}$ is anti-Hermitian, hence energy momentum tensors will be represented by anti-Hermitian three-form matrices. We give Einstein's equations for vacuum (V), vacuum with cosmological constant ($V + \Lambda$), and electrovacuum (EV) cases in the NP formalism for both component and matrix (compact) form.

(a) *Vacuum*: In the NP formalism we have $\phi_{ij} = \Lambda = 0$. The corresponding equations in compact notation are given as

$$R\tilde{\sigma} = 0. \quad (2.17)$$

TABLE I. Einstein's equations for vacuum, vacuum with cosmological constant, and electrovacuum in the Newman-Penrose formalism: component form and compact form.

Vacuum (V)	$\phi_{ij} = 0$	$\Lambda = 0$	$R\tilde{\sigma} = 0$
Vacuum with cosmological constant ($V + \Lambda$)	$\phi_{ij} = 0$	$\Lambda = \lambda_0$	$R\tilde{\sigma} + \lambda_0 \tilde{\sigma}\tilde{\sigma}\tilde{\sigma} = 0$
Electrovacuum (EV)	$\phi_{ij} = 2k\phi_i\bar{\phi}_j$	$\Lambda = 0$	$R\tilde{\sigma} - 2k\tilde{\sigma}\phi^\dagger\bar{\sigma}\phi\tilde{\sigma} = 0$
	ϕ_i 's given by Eqs. (2.20)		ϕ given by Eq. (2.24)
	Maxwell equations by Eqs. (2.21)		Maxwell equations by $d(\bar{\sigma}\phi\tilde{\sigma}) = 0$

(b) *Vacuum with cosmological constant (Einstein space)*: In the NP formalism Einstein equations are $\phi_{ij} = 0$ and $\Lambda = \lambda_0$, where λ_0 is a constant. The corresponding equations are given in compact form as

$$R\tilde{\sigma} + \lambda_0 \tilde{\sigma}\tilde{\sigma}\tilde{\sigma} = 0, \quad (2.18)$$

where λ_0 is a constant.

(c) *Electrovacuum*: In the NP formalism we have

$$\phi_{ij} = 2k\phi_i\bar{\phi}_j, \quad \Lambda = 0, \quad (2.19)$$

where k is the gravitational constant and the ϕ_i 's are given in terms of the Maxwell tensor $F_{\mu\nu}$ as

$$\phi_0 = F_{\mu\nu}l^\mu m^\nu, \quad (2.20a)$$

$$\phi_1 = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu), \quad (2.20b)$$

$$\phi_2 = F_{\mu\nu}\bar{m}^\mu n^\nu \quad (2.20c)$$

(recall that $l = l_\mu dx^\mu$, $n = n_\mu dx^\mu$, and $m = m_\mu dx^\mu$ in a coordinate basis), and the ϕ_i 's satisfy Maxwell's equations:

$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2, \quad (2.21a)$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2, \quad (2.21b)$$

$$\delta\phi_1 - \Delta\phi_0 = (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2, \quad (2.21c)$$

$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2, \quad (2.21d)$$

where $D\eta = l^\mu \partial_\mu \eta$, $\Delta\eta = n^\mu \partial_\mu \eta$, and $\delta\eta = m^\mu \partial_\mu \eta$ for a scalar η . For the purpose of formulating these equations in a compact form, we define the Maxwell two-form as

$$\mathcal{F} = F + i^*F = -\phi_1(ln - m\bar{m}) - \phi_0 n\bar{m} + \phi_2 lm, \quad (2.22)$$

where $F = F_{\mu\nu} dx^\mu dx^\nu$ and i^*F is the Hodge dual of F , i.e., $i^*F = \frac{1}{2}\epsilon_{\mu\nu}^{\alpha\beta} F_{\alpha\beta} dx^\mu dx^\nu$.

Then the Maxwell equations are simply $d\mathcal{F} = 0$. Now we can give electrovacuum Einstein equations in compact form as³⁰

$$R\tilde{\sigma} - 2k\tilde{\sigma}\phi^\dagger\bar{\sigma}\phi\tilde{\sigma} = 0, \quad (2.23)$$

where k is the gravitational constant, and

$$\phi = \begin{pmatrix} -\phi_1 & -\phi_2 \\ \phi_0 & \phi_1 \end{pmatrix}. \quad (2.24)$$

We write the Maxwell two-form as

$$I\mathcal{F} = \bar{\sigma}\phi\tilde{\sigma}, \quad (2.25a)$$

$$I\bar{\mathcal{F}} = -\tilde{\sigma}\phi^\dagger\bar{\sigma}, \quad (2.25b)$$

where I is the 2×2 identity matrix and $\bar{\mathcal{F}}$ is the complex conjugate of \mathcal{F} . Then Maxwell equations are

$$d(\bar{\sigma}\phi\tilde{\sigma}) = 0. \quad (2.26)$$

We summarize these results in Table I.

III. CARTAN IDEAL FOR EINSTEIN'S EQUATIONS

Cartan's geometric theory of partial differential equations (PDE's) essentially consists of the description of a PDE as a differential ideal I (equivalently, an exterior differential system) on a manifold N . Then an integral manifold of I will be the pair (M, f) , where M is a manifold and $f: M \rightarrow N$ is an embedding such that $f^*\alpha_i = 0$, for α_i in the ideal I . The differential system is said to be completely integrable if $f(M)$ is a submanifold of N . The terms, integral

submanifold or regular integral manifold, will also be used for such integral manifolds, and the term regular embedding will describe corresponding embeddings.

We recall that⁴ systems of PDE's can be represented as systems of homogeneous p -form equations (PDE's in the form of exterior differential equations) by possibly introducing new variables. We describe N to be a manifold with local coordinates consisting of all independent and dependent variables in the PDE's and auxiliary variables introduced in the previous step. Local coordinates of M will consist of independent variables. We then define the forms α_i on N such that their restriction to M gives the PDE's we started with, i.e., for the embedding $f: M \rightarrow N, f^*\alpha_i = 0$ gives our PDE's in the form of exterior differential equations. We remark that, if $f^*\alpha_i = 0$, then $f^*(\sum_i \xi_i \alpha_i) = 0$ also for forms ξ_i on N . Hence any form in the differential ideal $I = \{\alpha_i\}$ generated by the α_i vanish when restricted to M . Therefore we can conclude that it is rather the ideal generated by the α_i that represents the PDE's.

We note that different sets of generators may represent the same ideal: We define two exterior differential systems $\{\alpha_i\}$ and $\{\alpha'_i\}$ as algebraically equivalent if they generate the same ideal. Then $\{\alpha_i\}$ and $\{\alpha'_i\}$ will represent the same PDE, hence their integral manifolds will be (possibly different) solutions of this PDE.

The closure of an ideal $I = \{\alpha_i\}$ is an ideal $\bar{I} = \{\alpha_i, d\alpha_i\}$ obtained, by adjoining to I , exterior derivatives of its generators. An ideal and its closure have the same integral manifolds since $f^*d\alpha_i = df^*\alpha_i = 0$. An ideal is closed if it is algebraically equivalent to its closure (equivalently if $dI \subset I$). The study of differential systems is concerned mainly with regular integral manifolds of closed ideals.

In this study, our main interest will be the solution generation using Bäcklund transformations and we will not be interested in the existence and complete integrability problems. We only note that, for ideals generated by one-forms (Pfaff systems), the Frobenius theorem states that the system is completely integrable if and only if the corresponding ideal is closed.⁴ For higher-order systems we do not have such a complete result,^{29(b)} however, the Cartan-Kähler theorem provides (in the real analytic case) a method of construction for integral manifolds starting from lower-dimensional integral manifolds. We will now express the Einstein equations for various sources ($V, V + \Lambda, EV$) as a differential ideal I on a manifold N .

We start by describing local coordinates of M (space-time manifold to be embedded in N) as $x^\mu, \mu = 0, 1, 2, 3$. PDE's in the form of exterior differential equations, representing structure equations, are given by Eqs. (2.9a) and (2.9b), and their integrability conditions, i.e., Bianchi identities are given by (2.10a) and (2.10b). Also Eqs. (2.17), (2.18), and (2.23) represent, respectively, Einstein's equations for $V, V + \Lambda$, and EV fields. Dependent variables consist of the components of the metric (in fact coordinate components of the tetrad frame) of the connection of the curvature, and of the physical fields, for the nonvacuum case. They will constitute, together with x^μ , local coordinates of N . We will construct for $V, V + \Lambda$, and EV , a closed Cartan ideal with generators $\{\alpha_i\}$ such that, for $f: M \rightarrow N$ an embedding, $f^*\alpha_i = 0$ gives the corresponding structure and Einstein equations. To be precise, I is two sided, and includes complex conjugates of scalar forms, hence complex conjugates and Hermitian conjugates of matrix forms. We will give for each case ($V, V + \Lambda$, and EV) two algebraic equivalent sets of generators denoted by *generating set A* and *generating set B*.

Generating set A: This generating set defines connection and curvature forms and gives Einstein's equations. We will use this generating set to obtain a prolongation of I that will be used for constructing Bäcklund transformations.¹⁹

(a) *Vacuum:* $I = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger, \quad (3.1)$$

$$\alpha_2 = d\Gamma + \Gamma\Gamma - R, \quad (3.2)$$

$$\alpha_3 = R\bar{\sigma}, \quad (3.3)$$

$$\alpha_4 = dR - R\Gamma + \Gamma R. \quad (3.4)$$

We note that $\dim N = 58$ since we have 10, 24, and 20 variables to describe the metric, connection, and curvature, respectively. Also we remark that $d\alpha_1 = \alpha_3 + \alpha_3^\dagger$ and $d\alpha_2 = \alpha_4$, hence I is closed.

(b) *Vacuum with cosmological constant:* $I = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where α_1, α_2 , and α_4 are given by Eqs. (3.1), (3.2), and (3.4), respectively, and

$$\alpha_3 = R\bar{\sigma} + \lambda_0 \bar{\sigma}\bar{\sigma}\bar{\sigma}, \quad (3.5)$$

where λ_0 is a constant. The $\dim N$ is still 58. We remark that since $\bar{\sigma}\bar{\sigma}\bar{\sigma}$ is anti-Hermitian, $d\alpha_1 \subset I$, and, using Eq. (2.86), we obtain

$$d\bar{\sigma} = \bar{\alpha}_1 + \Gamma^\dagger \bar{\sigma} - \bar{\sigma}\Gamma. \quad (3.6)$$

TABLE II. The *generating set A* and *generating set B* for the Cartan ideal of Einstein's equations for vacuum, vacuum with cosmological constant, and electrovacuum.

	Vacuum	Vacuum with cosmological constant	Electrovacuum
<i>Generating set A</i>	$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_2 = d\Gamma + \Gamma\Gamma - R$ $\alpha_3 = R\bar{\sigma}$ $\alpha_4 = dR - R\Gamma + \Gamma R$	$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_2 = d\Gamma + \Gamma\Gamma - R$ $\alpha_3 = R\bar{\sigma} + \lambda_0 \bar{\sigma}\bar{\sigma}\bar{\sigma}$ $\alpha_4 = dR - R\Gamma + \Gamma R$	$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_2 = d\Gamma + \Gamma\Gamma - R$ $\alpha_3 = R\bar{\sigma} - 2k\bar{\sigma}\phi^\dagger\bar{\sigma}\phi\bar{\sigma}$ $\alpha_4 = dR - R\Gamma + \Gamma R$ $\alpha_5 = d(\bar{\sigma}\phi\bar{\sigma})$
<i>Generating set B</i>	$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_3 = (d\Gamma + \Gamma\Gamma)\bar{\sigma}$	$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_3 = (d\Gamma + \Gamma\Gamma)\bar{\sigma} + \lambda_0 \bar{\sigma}\bar{\sigma}\bar{\sigma}$	$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_3 = (d\Gamma + \Gamma\Gamma)\bar{\sigma} - 2k\bar{\sigma}\phi^\dagger\bar{\sigma}\phi\bar{\sigma}$ $\alpha_5 = d(\bar{\sigma}\phi\bar{\sigma})$

Then, it can be checked that $d\alpha_3 \subset I$, hence I is closed.

(c) *Electrovacuum*: $I = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, where α_1, α_2 , and α_4 are given by Eqs. (3.1), (3.2), and (3.4), respectively, and

$$\alpha_3 = R\bar{\sigma} - 2k\bar{\sigma}\phi^+ \bar{\sigma}\phi\bar{\sigma}, \quad (3.7)$$

$$\alpha_5 = d(\bar{\sigma}\phi\bar{\sigma}), \quad (3.8)$$

where k is the gravitational constant and ϕ is given by Eq. (2.25), and we note that $\bar{\sigma}\phi\bar{\sigma}$ is proportional to the identity matrix. For EV fields, $\dim N = 58 + 6 = 64$. Using Eqs. (3.6) and (3.8) it can be checked that $d\alpha_3 \subset I$, hence I is closed.

Generating set B: In this generating set, we omit the definition of the curvature, hence the Bianchi identities, then dimension of N is considerably reduced. This generating set will give a prolongation of I that has been proposed for inverse scattering problems.^{16,18,19}

(a) *Vacuum*: $I = \{\alpha_1, \alpha_3\}$, where α_1 is given by Eq. (3.1) and

$$\alpha_3 = (d\Gamma + \Gamma\Gamma)\bar{\sigma}, \quad (3.9)$$

than $\dim N = 38$, and again it can be seen that I is closed.

(b) *Vacuum with cosmological constant*: $I = \{\alpha_1, \alpha_3\}$, where α_1 is given by Eq. (3.1) and

$$\alpha_3 = (d\Gamma + \Gamma\Gamma)\bar{\sigma} + \lambda_0\bar{\sigma}\bar{\sigma}\bar{\sigma}, \quad (3.10)$$

also $\dim N = 38$ and I is closed.

(c) *Electrovacuum*: $I = \{\alpha_1, \alpha_3, \alpha_5\}$, where α_1 and α_5 are given by Eqs. (3.1) and (3.8) and

$$\alpha_3 = (d\Gamma + \Gamma\Gamma)\bar{\sigma} - 2k(\bar{\sigma}\phi^+ \bar{\sigma}\phi\bar{\sigma}). \quad (3.11)$$

In this case, $\dim N = 38 + 6 = 44$, and I is closed.

We present both sets of generators for $V, V + \Lambda$, and EV in Table II.

IV. PROLONGATION OF THE CARTAN IDEAL FOR EINSTEIN'S EQUATIONS

Prolongation of an ideal I in a manifold N is an ideal I' containing I , in a fiber bundle N' over N . If $\pi: N' \rightarrow N$ is the projection, we construct I' by lifting the generators $\{\alpha_i\}$ of I to N' and by adding new generators ω_A ; that is I' is generated by $\{\pi^*\alpha_i, \omega_A\}$. In this construction the ω_A 's are not of the form $\omega_A = d\theta_B$, for $\theta_B \in I'$. Then if $I' = \{\pi^*\alpha_i, \omega_A\}$ is closed, we have a nontrivial prolongation of I , and the PDE we started with is said to have a prolongation structure.^{5-7,13,16}

If $(M, f'), f': M \rightarrow N'$ is an integral manifold of I' , then $f'^*(\pi^*\alpha_i) = 0$ and $f'^*\omega_A = 0$. Therefore $(M, \pi \circ f')$ is an integral manifold of I with additional equations given by $f'^*\omega_A = 0$ also solved. Furthermore,¹⁰ for the nontrivial prolonged ideal I' constructed as above, the maximum-dimensional regular integral submanifolds of I and I' are the same. Therefore, if I is a completely integrable system, its prolongation I' is also completely integrable.⁴

In the prolongation process of an ideal representing a PDE, we are interested in finding additional forms that are linear in fiber variables. In these cases, they represent "associated linear equations" for original PDE's. In any case, prolongation forms are useful whenever they represent equations easier to solve than the original PDE's. Although the term "associated linear equation" has a special meaning in

inverse scattering problems, we will use the term "associated equations" to designate the equations represented by $f'^*\omega_A = 0$.

In the following sections we will give two different prolongations of the Cartan ideal constructed in Sec. III for $V, V + \Lambda$, and EV field equations. The first construction will give the linear equation proposed to be useful in the inverse scattering approach, the second will lead to Bäcklund transformations.

A. Prolonged Ideals using generating set B

In this section we construct a prolongation of I using *generating set B* given in Sec. III. In each case ($V, V + \Lambda, EV$) associated linear equations are formally the Rarita-Schwinger equations, and fiber variables admit a transformation that leave the associated equations and the ideal invariant. We give the prolonged ideals as follows.

(a) *Vacuum*: $I' = \{\alpha_1, \alpha_3, \omega\}$, where α_1, α_3 are given by Eqs. (3.1), (3.9), and

$$\omega = \bar{\sigma}(d + \Gamma)\Psi, \quad (4.1)$$

where Ψ is a complex vector one-form. Using Eqs. (3.1), (3.6), and (2.8b), we find

$$d\omega = \bar{\alpha}_1(d\Psi + \Gamma\Psi) + \bar{\alpha}_3^\dagger\Psi + \Gamma^\dagger\omega, \quad (4.2)$$

hence I' is closed.

The transformation

$$\Psi \rightarrow \Psi + (d + \Gamma)\varphi, \quad (4.3)$$

where φ is an arbitrary zero-form vector, leaves ω invariant (modulo I).

(b) *Vacuum with cosmological constant*: $I' = \{\alpha_1, \alpha_3, \omega_1, \omega_2\}$, where α_1, α_3 are given by Eqs. (3.1) and (3.10) and

$$\omega_1 = \bar{\sigma}[(d + \Gamma)\Psi_1 + \mu\bar{\sigma}\Psi_2], \quad (4.4a)$$

$$\omega_2 = \bar{\sigma}[(d - \Gamma^\dagger)\Psi_2 + \mu\bar{\sigma}\Psi_1], \quad (4.4b)$$

where $\mu^2 = \lambda_0$ and Ψ_1 and Ψ_2 are one-form vectors. Using Eqs. (3.1), (3.6) and (2.8b), we find

$$d\omega_1 = \bar{\alpha}_1(d\Psi_1 + \Gamma\Psi_1 + \mu\bar{\sigma}\Psi_2) - \mu\bar{\sigma}\alpha_1\Psi_2 + \bar{\alpha}_3^\dagger\Psi_1 + \Gamma^\dagger\omega_1 + \mu\bar{\sigma}\omega_2, \quad (4.5a)$$

$$d\omega_2 = \alpha_1(d\Psi_2 - \Gamma^\dagger\Psi_2 + \mu\bar{\sigma}\Psi_1) - \mu\bar{\sigma}\bar{\alpha}_1\Psi_1 + \alpha_3^\dagger\Psi_2 + \mu\bar{\sigma}\omega_1 - \Gamma\omega_2, \quad (4.5b)$$

hence I' is closed. It can also be checked that the transformation

$$\Psi_1 \rightarrow \Psi_1 + (d + \Gamma)\varphi_1 + \mu\bar{\sigma}\varphi_2, \quad (4.6a)$$

$$\Psi_2 \rightarrow \Psi_2 + (d - \Gamma^\dagger)\varphi_2 + \mu\bar{\sigma}\varphi_1, \quad (4.6b)$$

where φ_1, φ_2 are arbitrary zero-form vectors, leaves the ideal invariant.

(c) *Electrovacuum*: $I' = \{\alpha_1, \alpha_3, \alpha_5, \omega_1, \omega_2\}$, where α_1, α_3 , and α_5 are, respectively, given by Eqs. (3.1), (3.11), and (3.8), and

$$\omega_1 = \bar{\sigma}[(d + \Gamma)\Psi_1 + \mu\phi\bar{\sigma}\Psi_2], \quad (4.7a)$$

$$\omega_2 = \bar{\sigma}[(d - \Gamma^\dagger)\Psi_2 + \mu\phi^+\bar{\sigma}\Psi_1], \quad (4.7b)$$

where $\mu^2 = -2k$, Ψ_1 and Ψ_2 are one-form vectors. Using Eqs. (3.1), (3.6), (3.8), (2.8b), and (2.8c), we obtain

$$d\omega_1 = \tilde{\alpha}_1(d\Psi_1 + \Gamma\Psi_1) + \tilde{\alpha}_3^\dagger\Psi_1 + \mu\alpha_5\Psi_2 + \Gamma^\dagger\omega_1 + \mu\tilde{\sigma}\phi\omega_2, \quad (4.8a)$$

$$d\omega_2 = \alpha_1(d\Psi_2 - \Gamma^\dagger\Psi_2) + \alpha_3^\dagger\Psi_2 - \mu\alpha_5^\dagger\Psi_1 + \mu\tilde{\sigma}\phi^\dagger\omega_1 - \Gamma\omega_2, \quad (4.8b)$$

hence I' is a closed ideal. The transformation

$$\Psi_1 \rightarrow \Psi_1 + (d + \Gamma)\varphi_1 + \mu\phi\tilde{\sigma}\varphi_2, \quad (4.9a)$$

$$\Psi_2 \rightarrow \Psi_2 + (d - \Gamma^\dagger)\varphi_2 + \mu\phi^\dagger\tilde{\sigma}\varphi_1, \quad (4.9b)$$

leaves the ideal invariant.

Equation (4.1) was proposed as an associated linear equation for vacuum Einstein equations by Julia¹⁸ and Chinae,¹⁶ and is expected to be useful in finding solutions of Einstein's equations by inverse scattering technique. For space-times admitting symmetries, inverse scattering is reduced to two dimensions, and can be solved,¹¹⁻¹⁵ but in the general case, application of the inverse scattering technique to general relativity is an open problem. The results of this section are summarized in Table III.

B. Prolonged ideals using generating set A

Prolonged ideals described in this section will be used to find Bäcklund transformations. For notational convenience, we introduce the differential operator D as

$$D\eta = d\eta + \Gamma\eta + (-1)^p\eta\Gamma^\dagger, \quad (4.10)$$

for any p -form η . We give the prolonged ideals.

(a) *Vacuum*: $I' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \omega_1, \omega_2\}$, where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are given by Eqs. (3.1), (3.2), (3.3), and (3.4), respectively and

$$\omega_1 = D\tilde{t} + \omega\tilde{\sigma} + (\omega\tilde{\sigma})^\dagger, \quad (4.11)$$

$$\omega_2 = R\tilde{t} + D(\omega\tilde{\sigma}), \quad (4.12)$$

where \tilde{t} is a Hermitian and ω is a $\mathfrak{sl}(2, \mathbb{C})$ -valued (traceless) one-form. We then obtain

$$d\omega_1 = \alpha_2\tilde{t} + \tilde{t}\alpha_2^\dagger - \Gamma\omega_1 - \omega_1\Gamma^\dagger + \omega_2 + \omega_2^\dagger, \quad (4.13)$$

$$d\omega_2 = \alpha_2\omega\tilde{\sigma} + \omega\tilde{\sigma}\alpha_2^\dagger + \alpha_3\omega^\dagger + \omega\alpha_3^\dagger + \alpha_4\tilde{t} + R\omega_1 - \Gamma\omega_2 + \omega_2\Gamma^\dagger, \quad (4.14)$$

hence I' is closed.

(b) *Vacuum with cosmological constant*: $I' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \omega_1, \omega_2\}$, where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are given by Eqs. (3.1), (3.2), (3.5), and (3.4), respectively. The first prolongation form ω_1 will be the same as the one given for the vacuum case, i.e., ω_1 is given by Eq. (4.11). The second prolongation form will be obtained by adding a source term to the right-hand side of Eq. (4.12), i.e.,

$$\omega_2 = R\tilde{t} + D(\omega\tilde{\sigma}) - \chi_1. \quad (4.15)$$

We write χ_1 as

$$\chi_1 = -\mu_0 B_0 - \lambda_0 B_1, \quad (4.16)$$

where λ_0, μ_0 are constants, and

$$B_0 = \tilde{\sigma}\tilde{\sigma}\tilde{\sigma}, \quad (4.17)$$

$$B_1 = \tilde{\sigma}\tilde{\sigma}\tilde{t} + \tilde{\sigma}\tilde{t}\tilde{\sigma} + \tilde{t}\tilde{\sigma}\tilde{\sigma}. \quad (4.18)$$

We note that χ_1 is anti-Hermitian, hence Eq. (4.15) expresses that $d\omega_1 \subset I'$, and we need to check only $d\omega_2 \subset I'$. We will use the following forms of Eq. (4.11):

$$d\tilde{t} = \omega_1 - \Gamma\tilde{t} + \tilde{t}\Gamma^\dagger - \omega\tilde{\sigma} + \tilde{\sigma}\omega^\dagger, \quad (4.19)$$

$$d\tilde{t} = \tilde{\omega}_1 + \Gamma\tilde{t} - \tilde{t}\Gamma + \omega^\dagger\tilde{\sigma} - \tilde{\sigma}\omega. \quad (4.20)$$

In the following (i.e., for $V + \Lambda$ and EV) the calculation of exterior derivatives will be given, mod I' , since complete expressions become increasingly lengthy. We can obtain

$$dB_0 = -\Gamma\tilde{\sigma}\tilde{\sigma}\tilde{\sigma} + \tilde{\sigma}\tilde{\sigma}\tilde{\sigma}\Gamma^\dagger \pmod{I'},$$

$$dB_1 = -\Gamma(\tilde{\sigma}\tilde{\sigma}\tilde{t} + \tilde{\sigma}\tilde{t}\tilde{\sigma} + \tilde{t}\tilde{\sigma}\tilde{\sigma}) + (\tilde{\sigma}\tilde{\sigma}\tilde{t} + \tilde{\sigma}\tilde{t}\tilde{\sigma} + \tilde{t}\tilde{\sigma}\tilde{\sigma})\Gamma^\dagger - \omega\tilde{\sigma}\tilde{\sigma}\tilde{\sigma} + \tilde{\sigma}\tilde{\sigma}\tilde{\sigma}\omega^\dagger \pmod{I'}. \quad (4.21)$$

We rewrite them as

$$DB_0 = 0 \pmod{I'}, \quad (4.22)$$

$$DB_1 = (1/\lambda_0)[\omega R\tilde{\sigma} - R\tilde{\sigma}\omega^\dagger] \pmod{I'}. \quad (4.23)$$

Therefore,

$$D\chi_1 = -\omega R\tilde{\sigma} + R\tilde{\sigma}\omega^\dagger \pmod{I'}. \quad (4.24)$$

We now obtain $d\omega_2$ as

$$d\omega_2 = -D\chi_1 + R\tilde{\sigma}\omega^\dagger - \omega R\tilde{\sigma} \pmod{I'}, \quad (4.25)$$

and Eq. (4.24) implies that $d\omega_2 \subset I'$.

(c) *Electrovacuum*: $I = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \omega_1, \omega_2, \omega_3, \omega_4\}$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and ω_1 are given by Eqs. (3.1), (3.2), (3.7), (3.4), (3.8), and (4.11), respectively. We define

$$\omega_2 = R\tilde{t} + D(\omega\tilde{\sigma}) - \chi_1, \quad (4.26)$$

where

$$\chi_1 = A_0(\phi, \varphi) + A_0(\varphi, \phi) + A_0(\varphi, \varphi) + A_1(\phi, \phi), \quad (4.27)$$

where

$$A_0(u, v) = 2k[\tilde{\sigma}u^\dagger\tilde{\sigma}v\tilde{\sigma}], \quad (4.28)$$

$$A_1(u, v) = 2k[\tilde{\sigma}u^\dagger\tilde{\sigma}v\tilde{t} + \tilde{\sigma}u^\dagger v\tilde{\sigma} + \tilde{t}u^\dagger\tilde{\sigma}v\tilde{\sigma}]. \quad (4.29)$$

In Eqs. (4.28) and (4.29) and throughout Secs. IV and V, u and v replace either ϕ or φ , where ϕ is given by Eq. (2.24) and φ is a $\mathfrak{SL}(2, \mathbb{C})$ -valued zero-form (whose components are additional field variables).

TABLE III. Prolongation of the Cartan ideal of Einstein's equations using generating set B for vacuum, vacuum with cosmological constant, and electrovacuum.

Vacuum	Vacuum with cosmological constant	Electrovacuum
$\alpha_1 = d\tilde{\sigma} + \Gamma\tilde{\sigma} - \tilde{\sigma}\Gamma^\dagger$	$\alpha_1 = d\tilde{\sigma} + \Gamma\tilde{\sigma} - \tilde{\sigma}\Gamma^\dagger$	$\alpha_1 = d\tilde{\sigma} + \Gamma\tilde{\sigma} - \tilde{\sigma}\Gamma^\dagger$
$\alpha_3 = (d\Gamma + \Gamma\Gamma)\tilde{\sigma}$	$\alpha_3 = (d\Gamma + \Gamma\Gamma)\tilde{\sigma} + \lambda_0\tilde{\sigma}\tilde{\sigma}\tilde{\sigma}$	$\alpha_3 = (d\Gamma + \Gamma\Gamma)\tilde{\sigma} - 2k\tilde{\sigma}\phi^\dagger\tilde{\sigma}\phi\tilde{\sigma}$
$\omega = \tilde{\sigma}(d + \Gamma)\Psi$	$\omega_1 = \tilde{\sigma}[(d + \Gamma)\Psi_1 + \mu\tilde{\sigma}\Psi_2]$	$\alpha_5 = d(\tilde{\sigma}\phi\tilde{\sigma})$
	$\omega_2 = \tilde{\sigma}[(d - \Gamma^\dagger)\Psi_2 + \mu\tilde{\sigma}\Psi_1]$	$\omega_1 = \tilde{\sigma}[(d + \Gamma)\Psi_1 + \mu\phi\tilde{\sigma}\Psi_2]$
	where $\mu^2 = \lambda_0$	$\omega_2 = \tilde{\sigma}[(d - \Gamma^\dagger)\Psi_2 + \mu\phi^\dagger\tilde{\sigma}\Psi_1]$
		where $\mu^2 = -2k$

TABLE IV. Prolongation of the Cartan ideal of Einstein's equations using *generating set A* for vacuum, vacuum with cosmological constant and electrovacuum.

Vacuum	Vacuum with cosmological constant	Electrovacuum
$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_2 = d\bar{\Gamma} + \Gamma\bar{\Gamma} - R$ $\alpha_3 = R\bar{\sigma}$ $\alpha_4 = dR - R\bar{\Gamma} + \Gamma R$	$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_2 = d\bar{\Gamma} + \Gamma\bar{\Gamma} - R$ $\alpha_3 = R\bar{\sigma} + \lambda_0\bar{\sigma}\bar{\sigma}$ $\alpha_4 = dR - R\bar{\Gamma} + \Gamma R$	$\alpha_1 = d\bar{\sigma} + \Gamma\bar{\sigma} - \bar{\sigma}\Gamma^\dagger$ $\alpha_2 = d\bar{\Gamma} + \Gamma\bar{\Gamma} - R$ $\alpha_3 = R\bar{\sigma} - 2k\bar{\sigma}\phi^\dagger\bar{\sigma}\phi\bar{\sigma}$ $\alpha_4 = dR - R\bar{\Gamma} + \Gamma R$ $\alpha_5 = d(\bar{\sigma}\phi\bar{\sigma})$
$\omega_1 = D\bar{t} + \omega\bar{\sigma} + (\omega\bar{\sigma})^\dagger$ $\omega_2 = R\bar{t} + D(\omega\bar{\sigma})$	$\omega_1 = D\bar{t} + \omega\bar{\sigma} + (\omega\bar{\sigma})^\dagger$ $\omega_2 = R\bar{t} + D(\omega\bar{\sigma}) - \chi_1$, where $\chi_1 = -\mu_0 B_0 - \lambda_0 B_1$ $B_0 = \bar{\sigma}\bar{\sigma}\bar{\sigma}$ $B_1 = \bar{\sigma}\bar{\sigma}t + \bar{\sigma}t\bar{\sigma} + t\bar{\sigma}\bar{\sigma}$	$\omega_1 = D\bar{t} + \omega\bar{\sigma} + (\omega\bar{\sigma})^\dagger$ $\omega_2 = R\bar{t} + D(\omega\bar{\sigma}) - \chi_1$, where $\chi_1 = A_0(\phi, \varphi) + A_0(\varphi, \phi) + A_1(\phi, \phi)$ $A_0(u, v) = 2k(\bar{\sigma}u^\dagger\bar{\sigma}v\bar{\sigma})$ $A_1(u, v) = 2k(\bar{\sigma}u^\dagger\bar{\sigma}v\bar{\sigma} + \bar{\sigma}u^\dagger v\bar{\sigma} + iu^\dagger\bar{\sigma}v\bar{\sigma})$ $\omega_3 = d(\bar{\sigma}\phi^\dagger + i\phi\bar{\sigma})$ $\omega_4 = d(\bar{\sigma}\varphi\bar{\sigma})$

The remaining prolongation forms are

$$\omega_3 = d(\bar{\sigma}\phi^\dagger + i\phi\bar{\sigma}), \quad (4.30)$$

$$\omega_4 = d(\bar{\sigma}\varphi\bar{\sigma}). \quad (4.31)$$

Since ω_3 and ω_4 are exact and χ_1 is anti-Hermitian, we need to check only if $d\omega_2 \subset I'$. We also note that Eq. (4.26) and (4.15) differ only in the definition of χ_1 , therefore $d\omega_2$ will be in I' if (4.25) holds with χ_1 given as in (4.27). To obtain $dA_0(u, v)$ and $dA_1(u, v)$, we use $d(\bar{a}u^\dagger\bar{c}v\bar{b}) = \bar{a}u^\dagger d\bar{c}v\bar{b} + d(\bar{a}u^\dagger\bar{c})v\bar{b} - \bar{a}u^\dagger d(\bar{c}v\bar{b})$ (where a, b , and c replace either σ or t) and Eqs. (3.8), (4.30), and (4.31). We obtain

$$d[A_0(u, v)] = 2k[\bar{\sigma}u^\dagger d\bar{c}v\bar{\sigma}] \pmod{I'}, \quad (4.32)$$

$$d[A_1(\phi, \phi)] = 2k[\bar{\sigma}\phi^\dagger d\bar{\sigma}\phi\bar{t} + \bar{\sigma}\phi^\dagger d\bar{t}\phi\bar{\sigma} + t\phi^\dagger d\bar{\sigma}\phi\bar{\sigma}] \pmod{I'}. \quad (4.33)$$

Finally using Eqs. (3.6) and (4.20) (and also the fact that $\bar{\sigma}\phi\bar{\sigma}$ is proportional to the identity matrix), we write (4.32) and (4.33) as

$$DA_0(u, v) = 0 \pmod{I'}, \quad (4.34)$$

$$DA_1(\phi, \phi) = -\omega A_0(\phi, \phi) + A_0(\phi, \phi)\omega^\dagger \pmod{I'}, \quad (4.35)$$

and since $A_0(\phi, \phi) = R\bar{\sigma} \pmod{I'}$ we have verified that $d\omega_2 \subset I'$.

This completes the second set of prolonged ideals. The results of this section are summarized in Table IV. In the next section we will give a Bäcklund transform using these prolonged ideals.

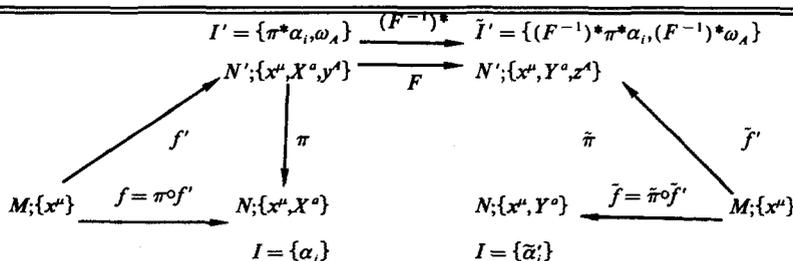
V. BÄCKLUND TRANSFORMATIONS OF EINSTEIN'S EQUATIONS

Bäcklund transformations are basically methods of generating new solutions from known solutions of a PDE. The prolonged ideals that we have constructed in Sec. IV B will be used to find such transformations. We describe the procedure as follows: I is an ideal on N (representing a PDE) generated by $\{\alpha_i\}$ and I' its prolongation on N' (fiber bundle over N , with projection π), with generators $\{\pi^*\alpha_i, \omega_A\}$. We start with a known solution, i.e., an integral submanifold (M, f) . Then the construction of I' ensures that (M, f') is an integral submanifold of I' with $f' = \pi \circ f$. We then look for a map $F: N' \rightarrow N'$ that induces a diffeomorphism of N , then F induces a (linear) mapping $(F^{-1})^*$ of forms in N . That is $\pi^*\alpha_i$ is mapped to $(F^{-1})^*\pi^*\alpha_i$, as a linear combination of α_i 's and ω_A 's. If we can choose a special form of F , such that the ideal generated by $\{(F^{-1})^*\pi^*\alpha_i\}$ is algebraically equivalent to I' , then $(M, F \circ f')$ will be a new solution for the PDE. Thus F is a Bäcklund transformation relating two solutions of the PDE. If furthermore F can be extended to a diffeomorphism on N' then we can describe $\tilde{I}' = \{(F^{-1})^*\pi^*\alpha_i, (F^{-1})^*\omega_A\}$ as prolonged from $(F^{-1})^*\pi^*\alpha_i$. This process is illustrated in Table V.

A. Construction of Bäcklund transformations for Einstein's equations

In order to obtain a Bäcklund transformation, we consider a map F on N' , such that $\bar{\sigma} \rightarrow \bar{\sigma} + t$, $\Gamma \rightarrow \Gamma + \omega$,

TABLE V. Bäcklund correspondence for Einstein's equations.



$R \rightarrow R + \rho, \phi \rightarrow \phi + \varphi$. Such a characterization ensures that F induces a diffeomorphism of N' . Then we denote $\tilde{\alpha}_i = (F^{-1})^* \pi^* \alpha_i$, and we can find explicit forms of $\tilde{\alpha}_i$ by simply replacing $\tilde{\sigma}, \Gamma, R, \phi$ in each α_i by their values under F . Therefore

$$\tilde{\alpha}_1 = \alpha_1 + D\tilde{t} + \omega\tilde{\sigma} + (\omega\tilde{\sigma})^\dagger + (\omega\tilde{t}) + (\omega\tilde{t})^\dagger, \quad (5.1)$$

$$\tilde{\alpha}_2 = \alpha_2 + d\omega + \omega\Gamma + \Gamma\omega + \omega\omega - \rho, \quad (5.2)$$

$$\tilde{\alpha}_4 = \alpha_4 + d\rho - R\omega + \omega R - \rho(\Gamma + \omega) + (\Gamma + \omega)\rho, \quad (5.3)$$

$$\tilde{\alpha}_3 = \alpha_3 + R\tilde{t} + D(\omega\tilde{\sigma}) + D(\omega\tilde{t}) + \omega\tilde{\sigma}\omega^\dagger + \omega\tilde{t}\omega^\dagger - \chi_1, \quad (5.4)$$

where

$$\chi_1 = \begin{cases} 0, & \text{for } V, \\ -(\lambda_0 + \mu_0)(\tilde{\sigma} + \tilde{t})(\tilde{\sigma} + \tilde{t})(\tilde{\sigma} + \tilde{t}) \\ \quad + \lambda_0\tilde{\sigma}\tilde{\sigma}\tilde{\sigma}, & \text{for } V + \Lambda, \end{cases} \quad (5.5)$$

$$\chi_1 = \begin{cases} 2k(\tilde{\sigma} + \tilde{t})(\phi^\dagger + \varphi^\dagger)(\tilde{\sigma} + \tilde{t})(\phi + \varphi)(\tilde{\sigma} + \tilde{t}) \\ \quad - 2k\tilde{\sigma}\phi^\dagger\tilde{\sigma}\phi\tilde{\sigma}, & \text{for EV;} \end{cases} \quad (5.6)$$

$$\tilde{\alpha}_5 = \alpha_5 + d[(\tilde{\sigma} + \tilde{t})(\phi + \varphi)(\tilde{\sigma} + \tilde{t})] - d[\tilde{\sigma}\phi\tilde{\sigma}]. \quad (5.7)$$

We note that $\tilde{\alpha}_2$ gives the definition of ρ and $\tilde{\alpha}_4$ gives an identity, then, to obtain algebraic equivalence, we have to express $\tilde{\alpha}_1, \tilde{\alpha}_3$, and $\tilde{\alpha}_5$ as a sum of α_i 's and ω_A 's, by equating remaining terms to 0. Comparing Eq. (5.1) with Eqs. (3.1) and (4.11) we obtain

$$\tilde{\alpha}_1 = \alpha_1 + \omega_1, \quad (5.8)$$

$$\omega\tilde{t} + (\omega\tilde{t})^\dagger = 0. \quad (5.9)$$

Now consider three cases.

(a) *Vacuum*: Comparing Eq. (5.4) with Eqs. (3.3) and (4.12), we have

$$\tilde{\alpha}_3 = \alpha_3 + \omega_2, \quad (5.10)$$

$$D(\omega\tilde{t}) + \omega\tilde{\sigma}\omega^\dagger + \omega\tilde{t}\omega^\dagger = 0. \quad (5.11)$$

(b) *Vacuum with cosmological constant*: Comparing Eqs. (5.4) and (5.6) with Eqs. (3.5) and (4.15), we obtain

$$\tilde{\alpha}_3 = \alpha_3 + \omega_2, \quad (5.12)$$

$$D(\omega\tilde{t}) + \omega\tilde{\sigma}\omega^\dagger + \omega\tilde{t}\omega^\dagger - \chi_2 = 0, \quad (5.13)$$

where

$$\chi_2 = -\lambda_0(B_2 + B_3) - \mu_0(B_1 + B_2 + B_3), \quad (5.14)$$

with

$$\begin{aligned} B_1 &= \tilde{\sigma}\tilde{\sigma}^\dagger + \tilde{\sigma}^\dagger\tilde{\sigma} + \tilde{t}\tilde{\sigma}, \\ B_2 &= \tilde{\sigma}^\dagger\tilde{t} + \tilde{t}\tilde{\sigma}^\dagger + \tilde{t}\tilde{t}, \\ B_3 &= \tilde{t}\tilde{t}. \end{aligned} \quad (5.15)$$

(c) *Electrovacuum*: Comparing Eqs. (5.4) and (5.8) with Eqs. (3.7), (4.26), (3.8), (4.30), and (4.31), we obtain

$$\tilde{\alpha}_3 = \alpha_3 + \omega_2, \quad (5.16a)$$

$$\tilde{\alpha}_5 = \alpha_5 + \omega_3 + \omega_4, \quad (5.16b)$$

$$D(\omega\tilde{t}) + \omega\tilde{\sigma}\omega^\dagger + \omega\tilde{t}\omega^\dagger - \chi_2 = 0, \quad (5.17a)$$

$$d(\tilde{\sigma}\phi\tilde{t} + \tilde{t}\phi\tilde{\sigma} + \tilde{t}(\phi + \varphi)\tilde{t}) = 0, \quad (5.17b)$$

where

$$\chi_2 = \sum_{i=1}^3 [A_i(\phi, \varphi) + A_i(\varphi, \phi) + A_i(\varphi, \varphi)]$$

$$+ \sum_{i=2}^3 A_i(\phi, \phi), \quad (5.18)$$

with

$$A_1(u, v) = 2k [\tilde{\sigma}u^\dagger\tilde{\sigma}v\tilde{t} + \tilde{\sigma}u^\dagger\tilde{t}v\tilde{\sigma} + \tilde{t}u^\dagger\tilde{\sigma}v\tilde{\sigma}],$$

$$A_2(u, v) = 2k [\tilde{\sigma}u^\dagger\tilde{t}v\tilde{t} + \tilde{t}u^\dagger\tilde{\sigma}v\tilde{t} + \tilde{t}u^\dagger\tilde{t}v\tilde{\sigma}], \quad (5.19)$$

$$A_3(u, v) = 2k [\tilde{t}u^\dagger\tilde{t}v\tilde{t}].$$

To obtain a Bäcklund transform for $V, V + \Lambda$, and EV, we propose to solve the equations

$$\omega\tilde{t} = 0, \quad (5.20)$$

$$\omega\tilde{\sigma}\omega^\dagger - \chi_2 = 0, \quad (5.21)$$

where χ_2 is 0 for V , and is given by Eqs. (5.14) and (5.18) for $V + \Lambda$ and EV, respectively.

B. A method of solution for $\omega\tilde{t} = 0, \omega\tilde{\sigma}\omega^\dagger = \chi_2$

We note that, since ω is defined by Eq. (4.12), it is unnecessary to consider it anymore as a fiber coordinate, as long as we solve for the components. Therefore, the first step will be, starting from an arbitrary parametrization of \tilde{t} , to solve for the components of ω from Eq. (4.11), i.e., the associated equation $f^*\omega_i = 0$; then to put restrictions either on the background or on the components of \tilde{t} such that $\omega\tilde{t} = 0$ and $\omega\tilde{\sigma}\omega^\dagger = \chi_2$, for the appropriate value of χ_2 . The equations for \tilde{t} will be first order, and in general, we will be interested in simple linear equations that are sufficient to solve $\omega\tilde{t} = 0, \omega\tilde{\sigma}\omega^\dagger = \chi_2$. We will give general expressions for ω with $\omega\tilde{t} = 0$. The expression for $\omega\tilde{\sigma}\omega^\dagger$ is cumbersome in the general case, but once ω is found, $\omega\tilde{\sigma}\omega^\dagger$ can be obtained by simple substitutions for each special type of transformation under consideration.

Solution of ω and the constraint equations completes the construction of the Bäcklund transform. But still we may have constraints on the background arising from algebraic or overdetermined equations in ω_2 .

We now solve ω from the associated equation $f^*\omega_1 = 0$, where $\omega_1 = D\tilde{t} + \omega\tilde{\sigma} - \tilde{\sigma}\omega^\dagger$: Starting with the following parametrization of \tilde{t} and as

$$\begin{aligned} \tilde{t} &= \begin{pmatrix} b_1 & -\bar{c}_1 \\ -c_1 & a_1 \end{pmatrix} l + \begin{pmatrix} b_2 & -\bar{c}_2 \\ -c_2 & a_2 \end{pmatrix} n \\ &+ \begin{pmatrix} d_2 & -d_4 \\ -d_3 & d_1 \end{pmatrix} m + \begin{pmatrix} \bar{d}_2 & -\bar{d}_3 \\ -\bar{d}_4 & \bar{d}_1 \end{pmatrix} \bar{m}, \end{aligned} \quad (5.22a)$$

$$\begin{aligned} \omega &= \begin{pmatrix} \omega_0^1 & \omega_1^1 \\ \omega_1^1 & -\omega_0^1 \end{pmatrix} l + \begin{pmatrix} \omega_0^2 & \omega_2^2 \\ \omega_2^2 & -\omega_0^2 \end{pmatrix} n \\ &+ \begin{pmatrix} \omega_0^3 & \omega_2^3 \\ \omega_2^3 & -\omega_0^3 \end{pmatrix} m + \begin{pmatrix} \omega_0^4 & \omega_2^4 \\ \omega_2^4 & -\omega_0^4 \end{pmatrix} \bar{m}, \end{aligned} \quad (5.22b)$$

we will solve ω from restriction of ω_1 [Eq. (4.11)] on M , i.e.,

$$d\tilde{t} + \Gamma\tilde{t} - \tilde{t}\Gamma^\dagger = -\omega\tilde{\sigma} + \tilde{\sigma}\omega^\dagger. \quad (5.23)$$

We write

$$d\tilde{t} + \Gamma\tilde{t} - \tilde{t}\Gamma^\dagger$$

$$\begin{aligned}
&= \begin{pmatrix} A_2 & -\bar{E}_1 \\ -E_1 & A_1 \end{pmatrix} ln + \begin{pmatrix} C_2 & -\bar{E}_3 \\ -E_2 & C_1 \end{pmatrix} lm \\
&+ \begin{pmatrix} \bar{C}_2 & -\bar{E}_2 \\ -E_3 & \bar{C}_1 \end{pmatrix} l\bar{m} + \begin{pmatrix} D_2 & -\bar{E}_5 \\ -E_4 & D_1 \end{pmatrix} nm \\
&+ \begin{pmatrix} \bar{D}_2 & -\bar{E}_4 \\ -E_5 & \bar{D}_1 \end{pmatrix} n\bar{m} + \begin{pmatrix} F_2 & \bar{E}_6 \\ -E_6 & F_1 \end{pmatrix} m\bar{m}, \quad (5.24)
\end{aligned}$$

and solve for the components of ω as

$$\begin{aligned}
\omega_0^1 &= \frac{1}{4}(-2A_2 + E_2 - \bar{E}_2 + F_2), \\
\omega_0^2 &= \frac{1}{4}(-2A_1 + E_4 - \bar{E}_4 + F_1), \\
\omega_0^3 &= \frac{1}{4}(-C_1 + D_2 + \bar{E}_1 + 2\bar{E}_6), \\
\omega_0^4 &= \frac{1}{4}(-\bar{C}_1 + \bar{D}_2 + E_1 - 2E_6), \\
\omega_1^1 &= \frac{1}{2}(\bar{C}_1 + \bar{D}_2 + E_1), \\
\omega_1^2 &= \bar{D}_1, \\
\omega_1^3 &= \frac{1}{2}(-E_4 - \bar{E}_4 + F_1), \quad (5.25) \\
\omega_1^4 &= -E_5, \\
\omega_2^1 &= C_2, \\
\omega_2^2 &= \frac{1}{2}(C_1 + D_2 - \bar{E}_1), \\
\omega_2^3 &= -\bar{E}_3, \\
\omega_2^4 &= -\frac{1}{2}(E_2 + \bar{E}_2 + F_2),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= -Da_1 + \Delta a_2 - 2(\gamma + \bar{\gamma})a_2 - \bar{\kappa}c_1 - \kappa\bar{c}_1 + \bar{\tau}c_2 \\
&\quad + \bar{\tau}\bar{c}_2 - (\tau + \bar{\pi})d_1 - (\bar{\tau} + \pi)\bar{d}_1, \\
A_2 &= -Db_1 + \Delta b_2 - 2(\epsilon + \bar{\epsilon})b_1 + \pi c_1 + \bar{\pi}\bar{c}_1 - \nu c_2 \\
&\quad - \bar{\nu}\bar{c}_2 - (\tau + \bar{\pi})d_2 - (\bar{\tau} + \pi)\bar{d}_2, \\
C_1 &= \bar{\delta}a_1 + \Delta d_1 - \bar{\tau}a_1 + \nu a_2 + \bar{\sigma}c_1 + \rho c_1 \\
&\quad + (\bar{\mu} - 2\bar{\gamma})d_1 + \bar{\lambda}\bar{d}_1 + (\tau + \bar{\tau})d_4, \\
C_2 &= \bar{\delta}b_1 + \Delta d_2 + [2(\alpha + \bar{\beta}) - \bar{\tau}]b_1 + \nu b_2 - \lambda c_1 \\
&\quad - \bar{\mu}\bar{c}_1 + (\bar{\mu} + 2\gamma)d_2 + \bar{\lambda}\bar{d}_2 - \nu d_3 - \bar{\nu}d_4, \\
D_1 &= \bar{\delta}a_2 + Dd_1 - \bar{\kappa}a_1 + [-2(\alpha + \bar{\beta}) + \pi]a_2 + \bar{\sigma}c_2 \\
&\quad + \rho\bar{c}_2 - (2\bar{\epsilon} + \rho)d_1 + \bar{\sigma}\bar{d}_1 + \bar{\kappa}d_3 + \kappa d_4, \\
D_2 &= \bar{\delta}b_2 + Dd_2 - \bar{\kappa}b_1 + \pi b_2 - \lambda c_2 - \bar{\mu}\bar{c}_2 \\
&\quad + (2\epsilon - \rho)d_2 - \bar{\sigma}\bar{d}_2 - \pi d_3 - \bar{\pi}d_4, \\
E_1 &= -Dc_1 + \Delta c_2 + \bar{\pi}a_1 - \bar{\nu}a_2 - \kappa b_1 + \tau b_2 \quad (5.26) \\
&\quad - 2\bar{\epsilon}c_1 - 2\gamma c_2 - (\tau + \bar{\pi})d_3 - (\bar{\tau} + \pi)\bar{d}_4, \\
E_2 &= \bar{\delta}c_1 + \Delta d_3 - \bar{\mu}a_1 + \rho b_1 + (2\bar{\beta} - \bar{\tau})c_1 \\
&\quad + \nu c_2 - \bar{\nu}d_1 + \tau d_2 + \bar{\mu}d_3 + \bar{\lambda}\bar{d}_4, \\
E_3 &= \delta c_1 + \Delta\bar{d}_4 - \bar{\lambda}a_1 + \sigma b_1 + (2\bar{\alpha} - \tau)c_1 + \bar{\nu}c_2 \\
&\quad - \bar{\nu}\bar{d}_1 + \bar{\tau}\bar{d}_2 + \bar{\lambda}d_3 + [\mu + 2(\bar{\gamma} - \gamma)]\bar{d}_4, \\
E_4 &= \bar{\delta}c_2 + Dd_3 - \bar{\mu}a_2 + \rho b_2 + (-2\alpha + \pi)c_2 \\
&\quad - \bar{\pi}d_1 + \kappa d_2 - \rho d_3 - \bar{\sigma}\bar{d}_4 - c_1\bar{\kappa}, \\
E_5 &= \delta c_2 + D\bar{d}_4 - \bar{\lambda}a_2 + \sigma b_2 + (-2\beta + \bar{\pi})c_2 - \bar{\pi}\bar{d}_1 \\
&\quad + \bar{\kappa}\bar{d}_2 - \sigma d_3 + [2(\bar{\epsilon} - \epsilon) - \bar{\rho}]\bar{d}_4 - c_1\kappa,
\end{aligned}$$

$$\begin{aligned}
E_6 &= \delta d_3 - \bar{\delta}\bar{d}_4 + (\rho - \bar{\rho})c_1 + (\mu - \bar{\mu})c_2 - \bar{\lambda}d_1 \\
&\quad + \bar{\mu}\bar{d}_1 + \sigma d_2 - \rho\bar{d}_2 + 2(\alpha - \bar{\beta})\bar{d}_4, \\
F_1 &= \delta d_1 - \bar{\delta}\bar{d}_1 + (\rho - \bar{\rho})a_1 + (\mu - \bar{\mu})a_2 - 2\bar{\alpha}d_1 \\
&\quad + 2\alpha\bar{d}_1 + \bar{\rho}d_3 - \rho\bar{d}_3 + \sigma d_4 - \bar{\sigma}\bar{d}_4, \\
F_2 &= \delta d_2 - \bar{\delta}\bar{d}_2 + (\rho - \bar{\rho})b_1 + (\mu - \bar{\mu})b_2 + 2\beta d_2 \\
&\quad - 2\bar{\beta}\bar{d}_2 - \mu d_3 + \bar{\mu}\bar{d}_3 - \bar{\lambda}d_4 + \bar{\lambda}\bar{d}_4.
\end{aligned}$$

We then write

$$\omega\bar{t} = \begin{pmatrix} -N & Q \\ P & -M \end{pmatrix},$$

where

$$\begin{aligned}
M &= [\omega_0^1 a_2 + \omega_1^1 \bar{c}_2 - \omega_0^2 a_1 - \omega_1^2 \bar{c}_1] ln \\
&\quad + [\omega_0^1 d_1 + \omega_1^1 d_4 - \omega_0^3 a_1 - \omega_1^3 \bar{c}_1] lm \\
&\quad + [\omega_0^1 \bar{d}_1 + \omega_1^1 \bar{d}_3 - \omega_0^4 a_1 - \omega_1^4 \bar{c}_1] l\bar{m} \\
&\quad + [\omega_0^2 d_1 + \omega_1^2 d_4 - \omega_0^3 a_2 - \omega_1^3 \bar{c}_2] nm \\
&\quad + [\omega_0^2 \bar{d}_1 + \omega_1^2 \bar{d}_3 - \omega_0^4 a_2 - \omega_1^4 \bar{c}_2] n\bar{m} \\
&\quad + [\omega_0^3 \bar{d}_1 + \omega_1^3 \bar{d}_3 - \omega_0^4 d_1 - \omega_1^4 d_4] m\bar{m}, \\
N &= [\omega_2^1 c_2 - \omega_0^1 b_2 - \omega_2^2 c_1 + \omega_0^2 b_1] ln \\
&\quad + [\omega_2^1 d_3 - \omega_0^1 d_2 - \omega_2^3 c_1 + \omega_0^3 b_1] lm \\
&\quad + [\omega_2^1 \bar{d}_4 - \omega_0^1 \bar{d}_2 - \omega_2^4 c_1 + \omega_0^4 b_1] l\bar{m} \\
&\quad + [\omega_2^2 d_3 - \omega_0^2 d_2 - \omega_2^3 c_2 + \omega_0^3 b_2] nm \\
&\quad + [\omega_2^2 \bar{d}_4 - \omega_0^2 \bar{d}_2 - \omega_2^4 c_2 + \omega_0^4 b_2] n\bar{m} \\
&\quad + [\omega_2^3 \bar{d}_4 - \omega_0^3 \bar{d}_2 - \omega_2^4 d_3 + \omega_0^4 d_2] m\bar{m}, \quad (5.27) \\
P &= [\omega_0^1 c_2 + \omega_1^1 b_2 - \omega_0^2 c_1 - \omega_1^2 b_1] ln \\
&\quad + [\omega_0^1 d_3 + \omega_1^1 d_2 - \omega_0^3 c_1 - \omega_1^3 b_1] lm \\
&\quad + [\omega_0^1 \bar{d}_4 + \omega_1^1 \bar{d}_2 - \omega_0^4 c_1 - \omega_1^4 b_1] l\bar{m} \\
&\quad + [\omega_0^2 d_3 + \omega_1^2 d_2 - \omega_0^3 c_2 - \omega_1^3 b_2] nm \\
&\quad + [\omega_0^2 \bar{d}_4 + \omega_1^2 \bar{d}_2 - \omega_0^4 c_2 - \omega_1^4 b_2] n\bar{m} \\
&\quad + [\omega_0^3 \bar{d}_4 + \omega_1^3 \bar{d}_2 - \omega_0^4 d_3 - \omega_1^4 d_2] m\bar{m}, \\
Q &= [\omega_2^1 a_2 - \omega_0^1 \bar{c}_2 - \omega_2^2 a_1 - \omega_0^2 c_1] ln \\
&\quad + [\omega_2^1 d_1 - \omega_0^1 d_4 - \omega_2^3 a_1 + \omega_0^3 \bar{c}_1] lm \\
&\quad + [\omega_2^1 \bar{d}_1 - \omega_0^1 \bar{d}_3 - \omega_2^4 a_1 + \omega_0^4 \bar{c}_1] l\bar{m} \\
&\quad + [\omega_2^2 d_1 - \omega_0^2 d_4 - \omega_2^3 a_2 + \omega_0^3 \bar{c}_2] nm \\
&\quad + [\omega_2^2 \bar{d}_1 - \omega_0^2 \bar{d}_3 - \omega_2^4 a_2 + \omega_0^4 \bar{c}_2] n\bar{m} \\
&\quad + [\omega_2^3 \bar{d}_1 - \omega_0^3 \bar{d}_3 - \omega_2^4 d_1 + \omega_0^4 d_4] m\bar{m}.
\end{aligned}$$

Then, to construct a Bäcklund transformation we start with a parametrization of \bar{t} as given in Eq. (5.22a), then insert these parameters in Eq. (5.26), and using (5.25) we determine the corresponding parametrization of ω , i.e., we have $\omega = \omega(\bar{t})$. We then insert these expressions in (5.27) to solve $\omega\bar{t} = 0$. The next section illustrates this method.

C. Examples for Bäcklund transformations of Einstein's equations

1. Example 1 (generalized Kerr–Schild transformation)

Generalized Kerr–Schild transformations constitute an important example for Bäcklund transformations of Einstein's equations. An extensive review of Kerr–Schild transformations can be found, for example, in Kramer *et al.* (see Ref. 23, Chap. 28). Generalized Kerr–Schild transformations are those with nonflat background, and they will be studied in detail in Sec. VI.

We choose \tilde{t} as

$$\tilde{t} = \begin{pmatrix} Vl & 0 \\ 0 & 0 \end{pmatrix}, \quad (5.28)$$

i.e., we let $l \rightarrow l$, $n \rightarrow n + Vl$, and $m \rightarrow m$. Then from Eq. (5.25), by taking $b_1 = V$, we obtain

$$\begin{aligned} \omega_0^1 &= \frac{1}{2}[DV + 2(\epsilon + \bar{\epsilon})V + (\rho - \bar{\rho})V], \\ \omega_0^1 &= \bar{\delta}V + [2(\alpha + \bar{\beta}) - \bar{\tau}]V, \\ \omega_2^3 &= -\bar{\sigma}V, \\ \omega_2^4 &= -\rho V, \end{aligned} \quad (5.29)$$

and we can see that, if $\kappa = 0$, then $\omega\tilde{t} = 0$, hence Eq. (5.10) is satisfied. Then we give

$$\omega\bar{\sigma}\omega^\dagger = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \quad (5.30)$$

where

$$\begin{aligned} S &= V\{-\frac{1}{2}(\rho + \bar{\rho})DV - (\epsilon + \bar{\epsilon})(\rho + \bar{\rho})V \\ &\quad + \frac{1}{2}(\rho^2 + \bar{\rho}^2)V - \sigma\bar{\sigma}V\}lm\bar{m}. \end{aligned} \quad (5.31)$$

We solve the remaining constraints for V , $V + \Lambda$, and EV as follows.

(a) *Vacuum*: We have

$$\omega\bar{\sigma}\omega^\dagger = 0. \quad (5.32)$$

Therefore, $\kappa = 0$ and

$$\begin{aligned} -\frac{1}{2}(\rho + \bar{\rho})DV - (\epsilon + \bar{\epsilon})(\rho + \bar{\rho})V \\ + \frac{1}{2}(\rho^2 + \bar{\rho}^2)V - \sigma\bar{\sigma}V = 0 \end{aligned} \quad (5.33)$$

solve the constraints.

(b) *Vacuum with cosmological constant*: We have

$$\omega\bar{\sigma}\omega^\dagger = \chi_2, \quad (5.34a)$$

where χ_2 is given by Eq. (5.20c). We note that $\bar{t}a\tilde{t} = \bar{t}a\tilde{t} = \bar{t}a\tilde{t} = \bar{t}a\tilde{t} \equiv 0$ for any a , and we obtain, from Eq. (5.14),

$$-\chi_2 = 3\mu_0 V \begin{pmatrix} lm\bar{m} & 0 \\ 0 & 0 \end{pmatrix}. \quad (5.34b)$$

Hence, using Eq. (5.30), we can see that $\kappa = 0$ and

$$\begin{aligned} -\frac{1}{2}(\rho + \bar{\rho})DV - (\epsilon + \bar{\epsilon})(\rho + \bar{\rho})V \\ + \frac{1}{2}(\rho^2 + \bar{\rho}^2)V - \sigma\bar{\sigma}V = -3\mu_0 V \end{aligned} \quad (5.35)$$

solve the constraints.

(c) *Electrovacuum*: The constraints are given by Eqs. (5.27a)–(5.27e). We start with

$$\omega\bar{\sigma}\omega^\dagger = \chi_2, \quad (5.36)$$

where χ_2 is given by Eq. (5.25c). We note that $A_2(\phi, \phi) \equiv 0$. Then

$$A_1(u, v) = 2kV \begin{pmatrix} -\bar{u}_1 v_1 lm\bar{m} & 0 \\ 0 & 0 \end{pmatrix} \quad (5.37)$$

and using (5.17b) we have

$$\begin{aligned} -\frac{1}{2}(\rho + \bar{\rho})DV - (\epsilon + \bar{\epsilon})(\rho + \bar{\rho})V \\ + \frac{1}{2}(\rho^2 + \bar{\rho}^2)V - \sigma\bar{\sigma}V \\ + 2k(\phi_1\bar{\varphi}_1 + \bar{\varphi}_1\phi_1 + \bar{\varphi}_1\varphi_1) = 0. \end{aligned} \quad (5.38)$$

Then we observe that

$$\bar{\sigma}u\tilde{t} + \bar{t}u\bar{\sigma} = -u_0 V lm\bar{m}, \quad (5.39)$$

hence we take $\varphi_0 = 0$ to satisfy Eq. (5.17b). Therefore, $\kappa = \phi_0 = \varphi_0 = 0$ and Eq. (5.38) solve the constraints.

2. Example 2

As a second example of Bäcklund transformations, we take the two-parameter transformation, which includes the GKS as a limit. This example is as follows:

$$\tilde{t} = \begin{pmatrix} V & -\bar{Z} \\ -Z & 0 \end{pmatrix} l, \quad (5.40)$$

i.e., we let $l \rightarrow l$, $n \rightarrow n + Vl$, and $m \rightarrow m + Zl$. For this example, we will construct only the Bäcklund correspondence, the solution of the associated equation will be presented elsewhere. To illustrate the Bäcklund construction process, we give the results in detail. We first set $b_1 = V$ and $c_1 = Z$. We also take $\kappa = 0$. Then from Eq. (5.26), we obtain the nonvanishing terms as

$$\begin{aligned} A_2 &= -DV - 2(\epsilon + \bar{\epsilon})V + \pi Z + \bar{\pi}\bar{Z}, \\ C_1 &= \rho\bar{Z} + \bar{\sigma}Z, \\ C_2 &= \bar{\delta}V + (2\alpha + 2\bar{\beta} - \bar{\tau})V - \lambda Z - \bar{\mu}\bar{Z}, \\ E_1 &= -DZ - 2\bar{\epsilon}Z, \\ E_2 &= \bar{\delta}Z + (2\bar{\beta} - \bar{\tau})Z + \rho V, \\ E_3 &= \delta Z + (2\bar{\alpha} - \tau)Z + \sigma V, \\ E_6 &= (\rho - \bar{\rho})Z. \end{aligned} \quad (5.41)$$

We already have $\omega_1^2 = \omega_1^3 = \omega_1^4 = \omega_0^2 = 0$. Inserting Eq. (5.41) in Eq. (5.27) we have

$$\begin{aligned} M &= 0, \\ N &= -\omega_2^2 Z l n + (-\omega_2^3 Z + \omega_0^3 V) l m \\ &\quad + (-\omega_2^4 Z + \omega_0^4 V) l \bar{m}, \\ P &= (-\omega_0^3 Z) l m + (-\omega_0^4 Z) l \bar{m}, \\ Q &= \omega_0^3 \bar{Z} l m + \omega_0^4 \bar{Z} l \bar{m}. \end{aligned} \quad (5.42)$$

We solve $\omega\tilde{t} = 0$ by taking (for $Z \neq 0$)

$$C_1 - \bar{E}_1 = E_2 = E_3 = 0, \quad \rho - \bar{\rho} = 0. \quad (5.43)$$

Therefore we have (for $Z \neq 0$)

$$D\bar{Z} + (2\epsilon + \rho)\bar{Z} + \bar{\sigma}Z = 0, \quad (5.44a)$$

$$\bar{\delta}Z + (2\bar{\beta} - \bar{\tau})Z + \rho V = 0, \quad (5.44b)$$

$$\delta Z + (2\bar{\alpha} - \tau)Z + \sigma V = 0. \quad (5.44c)$$

Nonvanishing components of ω are

$$\begin{aligned} \omega_0^1 &= \frac{1}{2}(DV + 2(\epsilon + \bar{\epsilon})V - \pi Z - \bar{\pi}\bar{Z}), \\ \omega_1^1 &= (\bar{\rho}Z + \sigma\bar{Z}), \end{aligned} \quad (5.45)$$

TABLE VI. Construction of Bäcklund correspondence for the examples given in Sec. V C.

	Vacuum	Vacuum with cosmological Constant	Electrovacuum
Example 1 GKS	$\kappa = 0$	$\kappa = 0$	$\kappa = 0, \phi_0 = 0$
$l \rightarrow \bar{l}$	$-\frac{1}{2}(\rho + \bar{\rho})DV - (\epsilon + \bar{\epsilon})(\rho + \bar{\rho})V$	$-\frac{1}{2}(\rho + \bar{\rho})DV - (\epsilon + \bar{\epsilon})(\rho + \bar{\rho})V$	$-\frac{1}{2}(\rho + \bar{\rho})DV - (\epsilon + \bar{\epsilon})(\rho + \bar{\rho})V$
$n \rightarrow n + VI$	$+\frac{1}{2}(\rho^2 + \bar{\rho}^2)V - \sigma\bar{\sigma}V = 0$	$+\frac{1}{2}(\rho^2 + \bar{\rho}^2)V - \sigma\bar{\sigma}V = -3\mu_0$	$+\frac{1}{2}(\rho^2 + \bar{\rho}^2)V - \sigma\bar{\sigma}V$
$m \rightarrow m$			$+ 2k(\phi_1 \bar{\varphi}_1 + \bar{\varphi}_1 \phi_1 + \varphi_1 \bar{\varphi}_1) = 0$
Example 2	$\kappa = 0, \rho - \bar{\rho} = 0$	$\kappa = 0, \bar{\rho} - \rho = 0, \mu_0 = 0$	
$l \rightarrow \bar{l}$			
$n \rightarrow n + VI$	$D\bar{Z} + (2\epsilon + \rho)\bar{Z} + \bar{\sigma}Z = 0$	$D\bar{Z} + (2\epsilon + \rho)\bar{Z} + \bar{\sigma}Z = 0$	
$m \rightarrow m + ZI$	$\delta Z + (2\beta - \bar{\tau})Z + \rho V = 0$	$\delta Z + (2\beta - \bar{\tau})Z + \rho V = 0$	
	$\delta Z + (2\bar{\alpha} - \tau)Z + \sigma V = 0$	$\delta Z + (2\bar{\alpha} - \tau)Z + \alpha V = 0$	

hence $\omega\bar{\sigma}\omega^\dagger \equiv 0$. We note that $\bar{l}a\bar{l} = \bar{l}a\bar{l} = \bar{l}a\bar{l} = \bar{l}a\bar{l} = 0$ for this transformation also with a an arbitrary zero-form matrix. Then for vacuum fields all constraints are satisfied identically, but for vacuum with cosmological constant we also need $\mu_0 = 0$ to construct a Bäcklund correspondence.

We summarize these results in Table VI.

VI. GENERALIZED KERR-SCHILD TRANSFORMATION

The generalized Kerr-Schild (GKS) transformation was given in Sec. V as an example of Bäcklund transformations of Einstein's equations. We started with the parametrization of \bar{l} given by Eq. (5.28) and we found that if $\kappa = \phi_0 = 0$ for the background (known solution) and if \bar{l} satisfies Eq. (5.33), (5.35), or (5.38) (depending on the source) we have a Bäcklund transformation. To find a new solution we have to solve the associated equations given in Table III for various cases.

As an alternative approach to the study of the GKS transformation, we use the Newman-Penrose (NP) formalism in component form (see Table I). Starting from transformation of the tetrad vectors of the form $l \rightarrow \bar{l}, n \rightarrow \bar{n}, m \rightarrow \bar{m}$, where

$$\bar{l} = l, \quad \bar{n} = n + VI, \quad \bar{m} = m, \quad (6.1)$$

it is possible to obtain the change in spin coefficients, components of the tracefree Ricci spinor, Weyl spinor, and scalar curvature in an arbitrary background (without assuming any symmetry or any simplifying choice of the tetrad²⁴). By a simple study of these expressions it can be seen that $\kappa = 0$ is a linearity requirement for the transformation of the tracefree Ricci spinor and the curvature scalar. Furthermore, it can be seen that Eqs. (5.33) and (5.35) are in fact one of the field equations for V and $V + \Lambda$, similarly Eq. (5.38) is a field equation for EV. Therefore if $\kappa = 0$, the solution of the associated equations in compact NP formalism is equivalent to solving the transformation equations in component form. The integrability of the solutions is guaranteed by construction of the prolonged ideal in the compact NP formalism.

In the following we will restrict ourselves to the case $\kappa = \phi_{00} = 0$ and give in Sec. VI A a brief review of Kerr-Schild metrics, the transformation of NP quantities, and some algebraic properties that follow. In Sec. VI B we give the mixed component of the Einstein tensor and we also ob-

serve that (since $l\mu$ is geodesic) the field equations for the new solution are linear in V . We then consider vacuum to vacuum and vacuum to electrovacuum transformations (in fact, we need only a part of the transformation equations to obtain the following results). The main result is that it is not possible to find algebraically general, asymptotically flat vacuum or electrovacuum metrics by applying the GKS transformation to a vacuum metric. This restriction is due to the fact that field equations are overdetermined and give a constraint on the background.

A. GKS transformation: Review and transformation properties

Kerr-Schild (KS) metrics³² have a special importance in the theory of general relativity. Einstein field equations constructed from these metrics are equivalent to the linearized field equations with flat background.³³ Most of the well-known metrics are in this class. These are the Schwarzschild, the Reissner-Nördstrom, the Kerr, the Kerr-Newman, the Vaidya, p - p waves, and the de Sitter-Schwarzschild metrics. All twisting metrics in this class are obtained from the nontwisting metrics by a complex translation.^{33,34} This is due to the fact that the field equations are Lorentz covariant and linear, hence the complexification generates the twisting solutions. In other words, the field equations are also invariant under complex Lorentz transformations.

The generalized Kerr-Schild (GKS) metrics are those with nonflat background. Until now, the space-times described by these metrics (GKS space-time) were considered only for vacuum³⁵ and pure radiation^{36,37} cases. Xanthopoulos³⁸ has shown that the vacuum field equations corresponding to the GKS metrics are equivalent to the linearized field equations in the background geometry. Taub³⁷ was interested in the pure radiation and cosmological solutions and using the GKS metrics he found some new solutions. In all the studies mentioned above the background metric is algebraically special. In this work we make no assumption on the background, and we find that there are in fact algebraic general backgrounds, admitting the GKS transformation. We now give the transformation of NP quantities.

The transformation of the spin coefficient can be either obtained by comparing with ω [Eq. (5.25)] or calculated

directly from the expressions for the derivatives of the tetrad frame (i.e., definition of connection), making use of

$$\tilde{D} = D, \quad \tilde{\Delta} = \Delta - VD, \quad \tilde{\delta} = \delta, \quad (6.2)$$

where (D, Δ, δ) and $(\tilde{D}, \tilde{\Delta}, \tilde{\delta})$ are derivative operators defined in Sec. II, for background and transformed spaces, respectively. In the following all symbols represent scalar quantities and $\tilde{}$ over a letter denotes the corresponding variable in the transformed space. We will also denote the background by (M, g) and the transformed space by (M, \tilde{g}) , where g and \tilde{g} denote the corresponding metrics. We recall that we restrict the background to the case $\kappa = 0$, $\phi_{00} = 0$, and we give the transformations of spin coefficients as

$$\begin{aligned} \tilde{\kappa} &= 0, \quad \tilde{\sigma} = \sigma, \quad \tilde{\rho} = \rho, \\ \tilde{\epsilon} &= \epsilon, \quad \tilde{\tau} = \tau, \quad \tilde{\pi} = \pi, \\ \tilde{\lambda} &= \lambda + \bar{\sigma}V, \quad \tilde{\mu} = \mu + \rho V, \\ \tilde{\alpha} &= \alpha, \quad \tilde{\beta} = \beta, \\ \tilde{\gamma} &= \gamma + \frac{1}{2}(D + 2\bar{\epsilon} + \rho - \bar{\rho})V, \\ \tilde{\nu} &= \nu + (\bar{\delta} + 2\alpha + 2\bar{\beta} - \tau - \pi)V. \end{aligned} \quad (6.3)$$

In Eq. (6.3) and from now on in this section, $\alpha, \beta, \gamma, \lambda, \mu, \nu, \rho, \sigma, \epsilon, \tau, \kappa, \pi$ will denote the NP spin coefficients.

Then the transformations of the tracefree Ricci spinor, curvature scalar, and Weyl spinor can be obtained from NP equations²² by direct substitution of Eq. (6.3) and making use of Eq. (6.1) as

$$\tilde{\phi}_{ij} = \phi_{ij} + S_{ij}, \quad i, j = 0, 1, 2, \quad (6.4)$$

$$\tilde{\Lambda} = \Lambda + S_{\Lambda}, \quad (6.5)$$

$$\tilde{\psi}_i = \psi_i + S_i, \quad i = 0, 1, 2, 3, 4, \quad (6.6)$$

where the S_{ij} 's, S_n , and S_i are

$$S_{00} = 0, \quad (6.7a)$$

$$S_{01} = 0, \quad (6.7b)$$

$$S_{02} = \sigma DV + \psi_0 V + \sigma[2(\epsilon + \bar{\epsilon}) + (\rho - \bar{\rho})]V, \quad (6.7c)$$

$$2S_{11} = \frac{1}{2}D^2V + \frac{3}{2}(\epsilon + \bar{\epsilon})DV + [D(\epsilon + \bar{\epsilon}) + (\epsilon + \bar{\epsilon})^2 + \sigma\bar{\sigma} + \rho\bar{\rho} - \rho^2 - \bar{\rho}^2]V, \quad (6.7d)$$

$$6S_{\Lambda} = -\frac{1}{2}D^2V - \frac{3}{2}(\epsilon + \bar{\epsilon})DV + (\rho + \bar{\rho})DV + [-D(\epsilon + \bar{\epsilon}) - (\epsilon + \bar{\epsilon})^2 + 2(\epsilon + \bar{\epsilon})(\rho + \bar{\rho}) + \sigma\bar{\sigma} - \rho\bar{\rho}]V, \quad (6.7e)$$

$$S_{12} = \frac{1}{2}\delta(D + 2\epsilon + 2\bar{\epsilon} + \rho - \rho)V + \sigma\bar{\delta}V + [\psi_1 + 2\sigma(\alpha + \bar{\beta}) - \sigma\bar{\tau} - \rho\tau + \frac{1}{2}(\bar{\alpha} + \beta - \tau)(D + 2\epsilon + 2\bar{\epsilon} + \rho - \bar{\rho})]V, \quad (6.7f)$$

$$S_{22} = -\Delta(\rho V) + [(\delta - \tau + \bar{\alpha} + 3\beta) \times (\bar{\delta} + 2\alpha + 2\bar{\beta} - \bar{\tau}) + \psi_2 - \mu(D + 2\epsilon + 2\bar{\epsilon} + \rho - \bar{\rho}) - \rho(\mu + \gamma + \bar{\gamma}) - \bar{\lambda}\bar{\sigma}]V, \quad (6.7g)$$

$$S_0 = 0, \quad (6.8a)$$

$$S_1 = 0, \quad (6.8b)$$

$$3S_2 = \frac{1}{2}(D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})$$

$$\times (D + 2\epsilon + 2\bar{\epsilon} + 3\rho - \bar{\rho})V - 2\sigma\bar{\sigma}V, \quad (6.8c)$$

$$S_3 = VD\alpha + \frac{1}{2}(\bar{\delta} + \bar{\beta} - \bar{\tau})(D + 2\bar{\epsilon} + \rho - \bar{\rho})V + (\rho + \epsilon)(\bar{\delta} + 2\alpha + 2\bar{\beta} - \bar{\tau} - \pi)V - (\tau + \beta)\bar{\sigma}V + \frac{1}{2}\alpha(D + 2\epsilon - \rho - \bar{\rho})V, \quad (6.8d)$$

$$S_4 = -\Delta(\bar{\sigma}V) + VD(\lambda + \bar{\sigma}V) + \bar{\delta}(\bar{\delta} + 2\alpha + 2\bar{\beta} - \bar{\tau} - \pi)V - \bar{\sigma}V(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) - (\lambda + \bar{\sigma}V)(D + 3\bar{\epsilon} - \epsilon + 3\rho - \bar{\rho})V + (3\bar{\alpha} + \beta + \pi - \bar{\tau}) \times (\bar{\delta} + 2\alpha + 2\bar{\beta} - \bar{\tau} - \pi)V. \quad (6.8e)$$

We now give a few results that can be obtained by a simple inspection of Eqs. (6.1)–(6.8).

(a) The transformation of the tetrad basis [Eq. (6.1)] shows that there is a subgroup of the local $SI(2, \mathbb{C})$ transformations in the background that also leave the transformed metric \tilde{g} invariant. This subgroup consists of null rotations around l^μ , and boosts in the l^μ - n^μ plane and spatial rotations in the m^μ - \bar{m}^μ plane.³¹ Also it can be seen that only the subgroup consisting of spatial rotations in m^μ - \bar{m}^μ plane commutes with the GKS transform.

(b) Since l^μ is a geodesic null vector in (M, g) [i.e., $l_{\mu;\nu}l^\nu = 0$, where a semicolon denotes the covariant derivative in (M, g)], and since $\kappa = l_{\mu;\nu}m^\mu l^\nu = 0$ is invariant under the GKS transformation, l_μ is also a geodesic null vector in (M, \tilde{g}) (i.e., $l_{\mu;\nu}l^\nu = 0$), where $|$ denotes the covariant derivative in (M, \tilde{g}) .

(c) If the tetrad frame $\tau^a = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ propagates parallelly along l^μ (i.e., $l^\mu \tau^a{}_{;\nu} = 0$) in (M, g) we have $\kappa = \epsilon = \pi = 0$. From the invariance of these spin coefficients we have that $\tilde{\tau}^a = (\tilde{l}^\mu, \tilde{n}^\mu, \tilde{m}^\mu, \tilde{\bar{m}}^\mu)$ propagates parallelly along l^μ in (M, \tilde{g}) also (i.e., $l^\mu \tilde{\tau}^a{}_{;\nu} = 0$).

(d) The optical scalars σ and ρ belonging to the null congruence with the tangent vector l^μ do not change.

(e) If l_μ is a principal null direction (pnd) of the Weyl tensor of (M, g) then $\psi_0 = 0$, and since l_μ is also geodesic, we have $\tilde{\psi}_0 = 0$, thus l_μ is also a pnd of the Weyl tensor in (M, \tilde{g}) .

(f) If the geodesic null vector l_μ is a pnd of (M, g) with multiplicity 1 (i.e., $\psi_0 = 0, \psi_1 \neq 0$, the space-time is called algebraically general), then it is a pnd with multiplicity 1 in (M, \tilde{g}) also. If l_μ is a pnd with multiplicity 2 (i.e., $\psi_0 = \psi_1 = \psi_2 \neq 0$, the space-time is called algebraically special), then it is a pnd with multiplicity at least 2 in (M, \tilde{g}) . In other words, if l_μ is a geodesic null vector and a pnd in (M, g) , then (M, \tilde{g}) will be algebraically general (special) if and only if (M, g) is algebraically general (special), but in the case (M, g) is algebraically special (M, \tilde{g}) need not be of the same type.

B. The field equations

Starting from the expression of the Einstein tensor given as

$$G_{\mu\nu} = 2\phi_{22}l_\mu l_\nu + 2\phi_{00}n_\mu n_\nu + 2\phi_{20}m_\mu m_\nu + 2\phi_{02}\bar{m}_\mu \bar{m}_\nu$$

$$\begin{aligned}
& + 2(\phi_{11} + 3\Lambda)(l_\mu n_\nu + n_\mu l_\nu) \\
& + 2(\phi_{11} - 3\Lambda)(m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu) \\
& - 2\phi_{21}(l_\mu m_\nu + m_\mu l_\nu) - 2\phi_{12}(l_\mu \bar{m}_\nu + \bar{m}_\mu l_\nu) \\
& - 2\phi_{10}(n_\mu m_\nu + m_\mu n_\nu) - 2\phi_{01}(n_\mu \bar{m}_\nu + \bar{m}_\mu n_\nu),
\end{aligned} \tag{6.9}$$

and using Eqs. (6.4)–(6.8), we obtain for the mixed component of the Einstein tensor

$$\begin{aligned}
\tilde{G}_\nu^\mu &= G_\nu^\mu - 2(S_{11} - 3S_\Lambda)\delta_\nu^\mu \\
& + 2[S_{22} - V^2(\phi_{00} + S_{00})]l^\mu l_\nu + 2S_{00}n^\mu n_\nu \\
& + 2S_{20}m^\mu m_\nu + 2S_{02}\bar{m}^\mu \bar{m}_\nu + 4S_{11}(l^\mu n_\nu + n^\mu l_\nu) \\
& - 2V(\phi_{00} + S_{00})(l^\mu n_\nu - n^\mu l_\nu) \\
& - 2S_{21}(l^\mu m_\nu + m^\mu l_\nu) \\
& + 2V(\phi_{10} + S_{10})(l^\mu m_\nu - m^\mu l_\nu) \\
& - 2S_{12}(l^\mu \bar{m}_\nu + \bar{m}^\mu l_\nu) \\
& + 2V(\phi_{01} + S_{01})(l^\mu \bar{m}_\nu - \bar{m}^\mu l_\nu) \\
& - 2S_{10}(n^\mu m_\nu + m^\mu n_\nu) - 2S_{01}(n^\mu \bar{m}_\nu + \bar{m}^\mu n_\nu).
\end{aligned} \tag{6.10}$$

Thus it can be seen that when l_μ is geodesic (note that $S_{22} - V^2\phi_{00}$ is linear for $\kappa = 0$) and $\phi_{00} = 0$, G_ν^μ is a linear functional of V . This is an important property that can be used for various purposes: If the background space-time is specified then the gravitational field equations become linear partial differential equations for V . Another consequence of this property follows: If a solution of the Einstein field equations can be put in the GKS form, i.e., $\tilde{g}_{\mu\nu} = g_{\mu\nu} + 2Vl_\mu l_\nu$, where $g_{\mu\nu}, V, l_\mu$ are all known and if (M, g) has a cyclic coordinate, then any complex translation generates a new solution of the Einstein field equations. These solutions may or may not be distinct.

Now we shall study the Einstein field equations, first assuming the background space-time is fixed. We recall that the transformation of the NP quantities are given for $\kappa = 0$ and $\phi_{00} = 0$. Also, for electrovacuum, ϕ_{00} implies $\phi_0 = 0$, hence $\phi_{01} = \phi_{02} = 0$. Since $S_{00} = 0$ we have $\phi_0 = 0$ therefore $S_{01} = S_{02} = 0$ (in fact $S_{01} \equiv 0$). Thus, in the cases under consideration, the backgrounds V and EV satisfy $\phi_{00} = \phi_{01} = \phi_{02} = 0$ and a first set of transformation equations are given by $S_{02} = S_\Lambda = 0$ (since $S_{00} = S_{01} = 0$). Also the equations giving the Bäcklund correspondence [Eq. (5.33) for V and Eq. (5.38) for EV] is equivalent to the $S_{11} + 3S_\Lambda$ term.

We rewrite the transformation of the tracefree Ricci spinor and curvature scalar for $V, V + \Lambda$, and EV as

$$S_{02} = 0, \tag{6.11a}$$

$$6S_\Lambda = 0, \tag{6.11b}$$

$$2S_{11} = 4k(\phi_1 \bar{\varphi}_1 + \varphi_1 \bar{\phi}_1 + \varphi_1 \bar{\varphi}_1), \tag{6.11c}$$

$$S_{12} = 2k(\phi_1 \bar{\varphi}_2 + \varphi_1 \bar{\phi}_2 + \varphi_1 \bar{\varphi}_2), \tag{6.11d}$$

$$S_{22} = 2k(\phi_2 \bar{\varphi}_2 + \varphi_2 \bar{\phi}_2 + \varphi_2 \bar{\varphi}_2). \tag{6.11e}$$

From (6.11) it can be seen that the solution of V depends crucially on $\rho + \bar{\rho}$ and σ , and the main compatibility

problem is in the solution of DV . For $\sigma = 0$, we have a consistent system, and the solution will depend on arbitrary constants only, unless $\rho + \bar{\rho} = 0$ (in this case ΔV is not specified and the solution will depend on an arbitrary function). But since $\kappa = 0$ and $\phi_{00} = \phi_{01} = \phi_{02} = 0$, then $\sigma = 0$ implies that background is algebraically special with $\psi_0 = \psi_1 = 0$ (Goldberg–Sachs theorem²³). Also it can be seen that $\rho + \bar{\rho} = 0$ implies $\sigma = 0$. In the following we will be interested in algebraic general backgrounds admitting the GKS transformation, hence we will take $\sigma \neq 0, \rho + \bar{\rho} \neq 0$. We analyze the solutions for two cases: (i) $S_{11} = 0$, and (ii) $S_{11} \neq 0$. The first case includes $V \rightarrow V, V \rightarrow \text{null EV}$ [$\phi_1 = 0$ (see Ref. 27)] transformations, and the second represent $V \rightarrow (\text{non-null})$ EV transformations.

(i) $S_{11} = 0$: We solve DV from Eq. (6.11c) as

$$DV = \omega_0 V, \tag{6.12}$$

where

$$\omega_0 = (\rho + \bar{\rho})^{-1}[\rho^2 + \bar{\rho}^2 - 2\sigma\bar{\sigma} - 2(\rho + \bar{\rho})(\epsilon + \bar{\epsilon})]. \tag{6.13}$$

Then $S_\Lambda = 0$ is satisfied, but from $S_{02} = 0$, we obtain the following constraint on the background:

$$\psi_0/2\sigma = (\rho + \bar{\rho})^{-1}(\sigma\bar{\sigma} - \rho^2). \tag{6.14}$$

(ii) $S_{11} \neq 0$: We solve DV from Eq. (6.11c). Then using $S_{02} = 0$, and $S_\Lambda = 0$, we obtain

$$\begin{aligned}
D[(\psi_0/\sigma) + 2\rho] \\
= [(\psi_0/\sigma) + 2\rho]^2 + (\epsilon + \bar{\epsilon})[(\psi_0/\sigma) + 2\rho].
\end{aligned} \tag{6.15}$$

Using Bianchi identities it can be seen that this condition is true whenever $S_{00} = S_{01} = S_{02} = S_0 = 0$.

Then V is the solution of

$$DV = \omega_0 V + u_0, \tag{6.16}$$

where ω_0 is given by Eq. (6.13), and

$$u_0 = 4k(\phi_1 \bar{\varphi}_1 + \bar{\varphi}_1 \phi_1 + \varphi_1 \bar{\varphi}_1). \tag{6.17}$$

The constraints (6.14) and (6.15) are satisfied in the cylindrical class of metrics described by Newman and Tamburino³⁹ and in stationary vacuum space-times discussed by Kota and Perjés.⁴⁰

We now give expressions for δV and ΔV . From Eq. (6.11d) we obtain

$$[(\rho^2 - \sigma\bar{\sigma})/(\rho + \bar{\rho})]\delta V + \sigma\bar{\sigma}V + \Omega V = S_{12}, \tag{6.18}$$

where

$$\begin{aligned}
\Omega = \psi_1 + (\delta + \bar{\alpha} + \beta - \tau)[(\rho^2 - \sigma\bar{\sigma})/(\rho + \bar{\rho})] \\
+ 2\sigma(\alpha + \bar{\beta}) - \sigma\bar{\tau} - \rho\tau
\end{aligned} \tag{6.19}$$

and

$$S_{12} = \begin{cases} 0, & \text{for } V \text{ and } V + \Lambda, \\ 2k(\phi_1 \bar{\varphi}_2 + \varphi_1 \bar{\phi}_2 + \varphi_1 \bar{\varphi}_2), & \text{for EV.} \end{cases} \tag{6.20}$$

Hence we can solve δV if

$$|(\rho^2 - \sigma\bar{\sigma})/(\rho + \bar{\rho})|^2 - \sigma\bar{\sigma} \neq 0, \tag{6.21}$$

as

$$\delta V = \omega_1 V + u_1, \tag{6.22}$$

where

$$\omega_1 = \left[\left| \frac{\rho^2 - \sigma\bar{\sigma}}{\rho + \bar{\rho}} \right|^2 - \sigma\bar{\sigma} \right]^{-1} \left[\sigma\bar{\Omega} - \frac{\bar{\rho}^2 - \sigma\bar{\sigma}}{\rho + \bar{\rho}} \Omega \right] \quad (6.23)$$

and

$$u_1 = \left[\left| \frac{\rho^2 - \sigma\bar{\sigma}}{\rho + \bar{\rho}} \right|^2 - \sigma\bar{\sigma} \right]^{-1} \left[-\sigma\bar{S}_{12} + \frac{\bar{\rho}^2 - \sigma\bar{\sigma}}{\rho + \bar{\rho}} S_{12} \right], \quad (6.24)$$

and, since $\rho + \bar{\rho} \neq 0$, we solve ΔV as

$$\Delta V = \omega_2 V - S_{22}, \quad (6.25)$$

where

$$\begin{aligned} \omega_2 = & (\rho + \bar{\rho})^{-1} [\delta(\alpha + \bar{\beta} - \bar{\tau} + \frac{1}{2}\bar{\omega}_1) \\ & + \bar{\delta}(\bar{\alpha} + \beta - \tau + \frac{1}{2}\omega_1) + \frac{1}{2}\omega_1(\bar{\omega}_1 + 3\alpha + 5\bar{\beta} - 2\bar{\tau}) \\ & + \frac{1}{2}\bar{\omega}_1(\omega_1 + 3\bar{\alpha} + 5\beta - 2\tau) + \psi_2 + \bar{\psi}_2 \\ & - (\rho + \bar{\rho})(\gamma + \bar{\gamma}) + \frac{1}{2}(\rho + \bar{\rho})^{-1}\mu(2\sigma\bar{\sigma} - 3\rho^2 + \bar{\rho}^2) \\ & + 1/2(\rho + \bar{\rho})^{-1}\bar{\mu}(2\sigma\bar{\sigma} - 3\bar{\rho}^2 + \rho^2) + (\alpha + \bar{\beta} - \bar{\tau}) \\ & \times (\bar{\alpha} + 3\beta) + 2\tau\bar{\tau} + (\bar{\alpha} + \beta - \tau)(\alpha + 3\bar{\beta}) \end{aligned} \quad (6.26)$$

and

$$S_{22} = \begin{cases} 0, & \text{for } V \text{ and } V + \Lambda, \\ 2k\{\phi_2\bar{\varphi}_2 + \varphi_2\phi_2 + \varphi_2\bar{\varphi}_2\}, & \text{for EV.} \end{cases} \quad (6.27)$$

Also ϕ_i and $\phi_i + \varphi_i$ satisfy Maxwell equations [Eqs. (2.21a)–(2.21d)] in (M, g) and (M, \bar{g}) , respectively. Then V , in general, can be solved from Eqs. (6.12), (6.18), and (6.25).

We now return to the constraint equations (6.14) and (6.16) and investigate the existence of asymptotically flat solutions. We recall that the behavior of these space-times has been studied by Newman and Unti²⁵ and by Exton, Newman, and Penrose²⁶ for vacuum and electrovacuum fields, respectively. Using their results we obtain the following theorems.

Theorem 1: If both (M, g) and (M, \bar{g}) are asymptotically flat vacuum space-times, then their Weyl tensors must be algebraically special. The GKS transform does not preserve algebraic generality transformations (see the next theorem for vacuum to electrovacuum transformations).

Proof: Following Newman and Unti²⁵ we can find a tetrad frame for which $\kappa = \pi = \epsilon = \rho = \bar{\rho} = \tau - \bar{\alpha} - \beta = 0$. In this tetrad Eq. (6.14) has the form

$$\psi_0/\sigma = (\sigma\bar{\sigma} - \sigma^2)/\rho. \quad (6.28)$$

We write the relevant NP equations as

$$D\rho = \rho^2 + \sigma\bar{\sigma}, \quad (6.29a)$$

$$D\sigma = 2\rho\sigma + \psi_0. \quad (6.29b)$$

Then Eq. (6.28) gives

$$D\sigma/\sigma = D\rho/\rho = \rho + (\sigma\bar{\sigma}/\rho). \quad (6.30)$$

Solving this equation, we have $\sigma = a\rho$, where $Da = 0$. Then (6.29a) gives

$$\rho(r) = [b - (1 + a\bar{a})r]^{-1} = O(r^{-1}), \quad (6.31)$$

where $Db = 0$, and $O(r^n)$ is the order of magnitude symbol. Thus

$$\sigma(r) = a[b - (1 + a\bar{a})r] = O(r^{-1}), \quad (6.32)$$

$$\psi_0(r) = a(a\bar{a} - 1)[b - (1 + a\bar{a})r]^{-2} = O(r^{-2}), \quad (6.33)$$

$$V(r) = V_0[b - (1 + a\bar{a})r]^{k_0}, \quad (6.34)$$

where $k_0 = (1 - a\bar{a})/(1 + a\bar{a})$ and $DV_0 = 0$. It is an asymptotically flat space-time²⁵ $\rho = O(r^{-1})$, $\sigma = O(r^{-2})$, and $\psi_0 = O(r^{-5})$, hence Eqs. (6.32) and (6.33), and shows clearly that (M, g) cannot be asymptotically flat, and, since ρ, σ, ψ_0 are invariant under the GKS transform (M, \bar{g}) will have the same behavior.

Theorem 2: If (M, g) is vacuum or electrovacuum with l^μ a principal null direction (pnd) of the Maxwell tensor ($\phi_0 = 0$), and (M, \bar{g}) a non-null ($\phi_1 \neq 0$) electrovacuum space-time, their Weyl tensors are algebraically special.

Proof: Asymptotic behavior of electrovacuum space-times are given in the same tetrad frame as in the previous theorem. Then using Eqs. (6.15) and (6.29b) we have

$$\frac{D\sigma}{\sigma} = \frac{D(\psi_0/\sigma + 2\rho)}{\psi_0/\sigma + 2\rho} = \frac{\psi_0}{\sigma} + 2\rho, \quad (6.35)$$

hence

$$\psi_0/\sigma + 2\rho = (b' - r)^{-1}, \quad Db' = 0, \quad (6.36a)$$

$$\sigma = a'(b' - r)^{-1}, \quad Da' = 0, \quad (6.36b)$$

but for asymptotically flat space-times,²⁶ $\sigma = O(r^{-2})$, also $\rho = O(r^{-1})$ and $\psi_0 = O(r^{-5})$. Therefore Eqs. (6.36a) and (6.36b) cannot be satisfied unless $\sigma = 0$, i.e., (M, g) hence (M, \bar{g}) are algebraically special.

VII. CONCLUSION

We expressed Einstein's equations for vacuum, vacuum with cosmological constant, and electrovacuum fields as a differential ideal using a compact Newman–Penrose formalism, and we gave two prolongations of this ideal generalizing previous works¹⁷ to vacuum with cosmological constant and electrovacuum fields. We obtained prolongations of these ideals and constructed Bäcklund transformations. Vacuum and vacuum with cosmological constant cases were also discussed in Ref. 19.

We also gave a method to obtain Bäcklund transformations. The generalized Kerr–Schild transformation is a well-known example to these transformations. Existence of other Bäcklund transformations will also be studied.

The transformations of Newman–Penrose quantities were given in Ref. 24. We present here the (linear) differential equations for the transformation parameter for each case discussed above and show that algebraically general asymptotically flat vacuum and electrovacuum solutions cannot be obtained (starting from a vacuum solution).

We finally remark that when the space-time admits two Killing vectors, it is known that Einstein's equations constitute a completely integrable system. For space-times without symmetry, the Rarita–Schwinger equation^{16,18} was proposed as an associated linear equation for Einstein's equations, therefore in the case of two Killing vectors this equation should reduce to the Belinski–Zakharov system. This problem is currently studied.

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