Gen Relativ Gravit (2009) 41:31–38
DOI 10.1007/s10714-008-0648-y

RESEARCH ARTICLE

# Gödel type metrics in Einstein–aether theory

**Metin Gürses**

Received: 26 February 2008 / Accepted: 22 April 2008 / Published online: 9 May 2008


**Abstract** Aether theory is introduced to implement the violation of the Lorentz invariance in general relativity. For this purpose a unit timelike vector field is introduced to the theory in addition to the metric tensor. Aether theory contains four free parameters which satisfy some inequalities in order that the theory to be consistent with the observations. We show that the Gödel type of metrics of general relativity are also exact solutions of the Einstein–aether theory. The only field equations are the 3D Maxwell field equations and the parameters are left free except $c_1 - c_3 = 1$.

**Keywords** Gödel type metrics · Einstein–Maxwell–Dust field equations · Aether theory

## 1 Introduction

Noncommutativity of local coordinates seems to be an important implication of the string theory. Such models are considered in quantum field theory and it is observed that Lorentz invariance is broken due to the additional terms coming from the noncommutativity [1,2]. Lorentz violating theories may lead to some new effects in astrophysics and cosmology [3].

In order to include Lorentz symmetry violating terms in gravitation theories, apart from some noncommutative gravity models, one may also consider existence of preferred frames. This can be achieved admitting a unit timelike vector field in addition to the metric tensor of spacetime. Such a timelike vector implies a preferred direction at each point of spacetime. Here the unit timelike vector field is called the aether and

M. Gürses (✉)
Department of Mathematics, Faculty of Sciences, Bilkent University, 06800 Ankara, Turkey
e-mail: gurses@fen.bilkent.edu.tr

the theory coupling the metric and unit timelike vector is called the Einstein–aether theory. In the last decade there is an increasing interest in the aether theory [4]–[10].

Let $u^\mu$ be unit timelike vector ($u^\mu\, u_\mu = -1$) and let a four rank tensor $K^{\mu\nu}{}_{\alpha\beta}$ be given by

$$K^{\mu\nu}{}_{\alpha\beta} = c_1\, g^{\mu\nu}\, g_{\alpha\beta} + c_2\, \delta^\mu_\alpha\, \delta^\nu_\beta + c_3\, \delta^\mu_\beta\, \delta^\nu_\alpha - c_4\, u^\mu\, u^\nu\, g_{\alpha\beta}, \tag{1}$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are the constants of the theory. The action of the theory is given as $\mathcal{L}$

$$I = \frac{1}{16\pi G} \int \sqrt{-g}\, \mathcal{L}\, d^4\, x, \tag{2}$$

where

$$\mathcal{L} = R - K^{\alpha\beta}{}_{\mu\nu}\, \nabla_\alpha\, u^\mu\, \nabla_\beta\, u^\nu + \lambda\, \left(u^\mu\, u_\mu + 1\right). \tag{3}$$

We define a second rank tensor $J^\mu{}_\nu$ as

$$J^\mu{}_\nu = K^{\mu\alpha}{}_{\nu\beta}\, \nabla_\alpha\, u^\beta. \tag{4}$$

Then the field equations of the aether theory are given by

$$\begin{aligned} G_{\mu\nu} = {} & \nabla_\alpha\, \left[J^\alpha{}_{(\mu}\, u_{\nu)} - J_{(\mu}{}^\alpha\, u_{\nu)} + J_{(\mu\nu)}\, u^\alpha\right] + c_1\, \left(\nabla_\mu\, u_\alpha\, \nabla_\nu\, u^\alpha - \nabla_\alpha\, u_\mu\, \nabla^\alpha\, u_\nu\right) \\ & + c_4\, \dot{u}_\mu\, \dot{u}_\nu + \lambda u_\mu\, u_\nu - \frac{1}{2} L\, g_{\mu\nu}, \end{aligned} \tag{5}$$

$$c_4\, \dot{u}^\alpha\, \nabla_\mu\, u_\alpha + \nabla_\alpha\, J^\alpha{}_\mu + \lambda u_\mu = 0, \tag{6}$$

$$u^\mu\, u_\mu = -1, \tag{7}$$

where $\dot{u}^\mu = u^\alpha\, \nabla_\alpha\, u^\mu$ and

$$\lambda = c_4\, \dot{u}^\alpha\, \dot{u}_\alpha + u^\alpha\, \nabla_\beta\, J^\beta{}_\alpha, \tag{8}$$

$$L = K^{\mu\nu}{}_{\alpha\beta}\, (\nabla_\mu\, u^\alpha)\, (\nabla_\nu\, u^\beta). \tag{9}$$

The action given above is invariant under the redefinition [11,12]

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - (1 - B) u_\mu\, u_\nu, \tag{10}$$

$$\tilde{u}^\mu = \frac{1}{\sqrt{B}}\, u^\mu \tag{11}$$

where $B$ is a positive constant. Then the parameters transform as

$$c_1 = \frac{1}{2B}\left[(1+B^2)\tilde{c}_1 + (1-B^2)\,\tilde{c}_3 - 1 + B^2\right], \tag{12}$$

$$c_2 = \frac{1}{B}\,(\tilde{c}_2 + 1 - B)\,, \tag{13}$$

$$c_3 = \frac{1}{2B}\left[(1-B^2)\tilde{c}_1 + (1+B^2)\tilde{c}_3 - 1 + B^2\right], \tag{14}$$

$$c_4 = \tilde{c}_4 - \frac{1}{2B}\left[(1-B^2)\tilde{c}_1 + (1-B^2)\tilde{c}_3 - 1 + B^2\right]. \tag{15}$$

These transformations imply that letting for instance all $\tilde{c}_i$, $i = 1, 2, 3, 4$ to vanish then we have a special type of aether theory. This means that $c_i$s are all related. $\tilde{g}_{\mu\nu}$ is the metric of a vacuum spacetime and $g_{\mu\nu}$ is the metric of the aether theory with these special $c_i$s.

A special case is the Einstein–Maxwell theory with dust distribution (no pressure) [7] (see also [13,14]). Let $c_2 = c_4 = 0$ and $c_3 = -c_1$. Then $J^{\mu}{}_{\nu} = c_1\, F^{\mu}{}_{\nu}$ and the action above becomes

$$I = \frac{1}{16\pi G}\int \sqrt{-g}\,[R - c_1\, F^2 + \lambda\,(u^{\mu}\, u_{\mu} + 1)]\, d^4\, x \tag{16}$$

with the filed equations

$$G_{\mu\nu} = c_1\, T_{\mu\nu} + \lambda u_{\mu}\, u_{\nu}, \tag{17}$$

$$\nabla_{\mu}\, F^{\mu\nu} = \frac{\lambda}{c_1}\, u^{\nu}, \tag{18}$$

$$u^{\mu}\, u_{\mu} = -1, \tag{19}$$

where $F_{\mu\nu} = \nabla_{\nu}\, u_{\mu} - \nabla_{\mu}\, u_{\nu}$ and $T_{\mu\nu}$ is the energy momentum tensor of the filed $F_{\mu\nu}$. This theory differs from the Einstein Maxwell theory due to the last equation (19) which brakes the gauge invariance of the theory. A generalization of the above special theory is given in [14], called TeVsS. This theory contains also a scalar (dilaton) field coupling to the unit timelike vector field and the metric tensor.

The parameters $c_1$, $c_2$, $c_3$ and $c_4$ are not so free. They satisfy some inequalities in order that the aether theory to be compatible with some observations [6,10].

1. It is shown that this theory has the same PPN parameters as those of general relativity if

$$c_2 = \frac{-2c_1^2 - c_1 c_3 + c_3^2}{3c_1}, \quad c_4 = -\frac{c_3^2}{c_1}. \tag{20}$$

2. In the slow motion limit of the theory the constant playing the role of Newton's constant is

$$G_N = G\left(1 - \frac{c_1 + c_4}{2}\right)^{-1} \tag{21}$$

where $G$ is Newton's constant.
3. In Friedman–Robertson–Walker type of cosmological models, the theory admits the cosmological gravitational constant

$$G_{\text{cosmos}} = G\left(1 + \frac{c_+ + 3c_2}{2}\right)^{-1} \tag{22}$$

where $c_+ = c_1 + c_3$.
4. Primordial abundance of $^4$He gives

$$\left|\frac{G_{\text{cosmo}}}{G_N} - 1\right| < \frac{1}{8}. \tag{23}$$

5. From the maximum mass of the neutron stars

$$c_1 + c_4 \leq 0.5 - 1.6. \tag{24}$$

6. Stability against linear perturbations in Minkowski background we have

$$0 < c_1 + c_3 < 1, \quad 0 < c_1 - c_3 < \frac{c_+}{3(1 - c_+)}. \tag{25}$$

For example, when $c_1 - c_3 = 1$ the above constraints are all satisfied where $\frac{G_{\text{cosmo}}}{G_N} = 1$, and $\frac{7}{8} < c_1 < 1$, $-\frac{1}{8} < c_3 < 0$.

In this work we first give the Gödel type metrics in general relativity by presenting a short summary of [15]. We show that the Gödel type metrics form an exact solution of the Einstein field equations with a charged dust distribution. The only field equations to be solved are the three dimensional Euclidean Maxwell equations. Next we show that Gödel type metrics solve also the field equations of the Einstein–aether theory. The only remaining equations are again the three dimensional Euclidean Maxwell equations corresponding the unit timelike vector field and the constraint $c_1 - c_3 = 1$. It seems that this constraint is compatible with the bounds of the parameters of Einstein-aether theory.

## 2 Gödel type metrics in general relativity

Let $u^\mu = -\delta_0^\mu$ be a timelike vector with $u_0 = 1$ in 4D spacetime $M$ and $h_{\mu\nu}$ be a constant tensor ($\partial_\alpha h_{\mu\nu} = 0$) such that $u^\mu h_{\mu\nu} = 0$. Gödel type of metrics are defined by [15]

$$g_{\mu\nu} = h_{\mu\nu} - u_\mu \, u_\nu. \tag{26}$$

It is easy to show that $u^\mu$ is also a Killing vector of the spacetime geometry $(M, g)$. Then we can define an antisymmetric tensor $f_{\mu\nu}$ as

$$f_{\alpha\beta} = u_{\beta,\alpha} - u_{\alpha,\beta} = 2u_{\beta;\alpha} \tag{27}$$

where semicolon denotes covariant derivative with respect to the Christoffel symbol. The Christoffel symbol corresponding to the metric (26) is

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\left(u_\alpha \, f^\mu{}_\beta + u_\beta \, f^\mu{}_\alpha\right) - \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) u^\mu. \tag{28}$$

It is easy to show that

$$u^\alpha \, \partial_\alpha \, u_\beta = 0, \quad u^\alpha \, f_{\alpha\beta} = 0. \tag{29}$$

Then

$$\dot{u}^\mu = u^\alpha \, u^\mu{}_{;\alpha} = 0 \tag{30}$$

It is now straightforward to show that the Einstein tensor becomes

$$G_{\mu\nu} = \frac{1}{2} \, T^f_{\mu\nu} + \frac{1}{4} f^2 \, u_\mu \, u_\nu \tag{31}$$

provided $f$ satisfies the equation

$$\partial_\alpha \, f_\mu{}^\alpha = 0 \tag{32}$$

where $T^f_{\mu\nu}$ is the Maxwell energy momentum tensor for the antisymmetric tensor $f_{\mu\nu}$

$$T^f_{\mu\nu} = f_{\mu\alpha} \, f_\nu{}^\alpha - \frac{1}{4} \, f^2 \, g_{\mu\nu}$$

where $f^2 = f^{\alpha\beta} \, f_{\alpha\beta}$. Maxwell's equations (32) can also be written as

$$\nabla_\alpha \, f^{\alpha\mu} = \frac{1}{2} \, f^2 \, u^\mu \tag{33}$$

Hence Gödel type metrics (26) satisfy the Einstein field equations with charged dust distributions. The only field equations are the Maxwell equations (32) or (33) which may further be reduced to a more simpler form

$$\partial_i \, f_{ij} = 0 \tag{34}$$

Hence there is no electric field ($u^{\mu} f_{\mu i} = f_{0i} = 0$), only the magnetic field exists. Maxwell equations (43) are in Euclidean 3D. We exhibited some solutions of this equation and hence explicit Gödel type metrics in [15]. All such spacetimes contain closed timelike and closed null curves. Some examples of these metrics are are given as follows [15]:

(a) Let $u_{\mu}\, dx^{\mu} = dt + b\,(x^2\, dx^1 - x^1\, dx^2)$. Then $f_{ij}\, dx^i \wedge dx^j = 2b\, dx^1 \wedge dx^2$. Hence (43) is satisfied identically. The the metric and the unit timelike vector $u_{\mu}$ in cylindrical coordinates are given by

$$ds^2 = -(dt - b\rho^2 d\phi)^2 + d\rho^2 + \rho^2\, d\phi^2 + dz^2, \tag{35}$$

$$u_{\mu}\, dx^{\mu} = dt - b\rho^2\, d\phi, \tag{36}$$

where $b$ is an arbitrary constant.

(b) Let $u_i = \psi\, \delta_i^3$ where $\psi$ is an harmonic function of $x^1$ and $x^2$ ($\nabla^2\, \psi = 0$). Then the Maxwell equations (43) are satisfied identically. Metric tensor and the unit timelike vector field $u_{\mu}$ are given by

$$ds^2 = -(dt - \psi(\rho, \phi) dz)^2 + d\rho^2 + \rho^2\, d\phi^2 + dz^2, \tag{37}$$

$$u_{\mu}\, dx^{\mu} = dt + \psi(\rho, \phi) dz. \tag{38}$$

We have also considered the Gödel type metrics when $u_0$ is not a constant [16]. In this case the proposed metric constitute an exact solution of several theories with a dilaton field.

## 3 Gödel type metrics in aether theory

We assume that the metric $g_{\mu\nu}$ and the timelike four vector $u^{\mu}$ are the Gödel type metric and the timelike vector defined in the previous section. With these assumptions we find that

$$J^{\mu}{}_{\nu} = \frac{1}{2}\,(c_1 - c_3)\, f^{\mu}{}_{\nu}, \tag{39}$$

$$\lambda = -\frac{1}{4}\,(c_1 - c_3)\, f^2, \tag{40}$$

$$L = \frac{1}{4}\,(c_1 - c_3)\, f^2 \tag{41}$$

Then the Einstein field equations (5) becomes

$$G_{\mu\nu} = (c_1 - c_3)\left[\frac{1}{2}\, T^f_{\mu\nu} + \frac{1}{4} f^2\, u_{\mu}\, u_{\nu}\right] \tag{42}$$

and the aether equation (6) reduces to

$$\nabla_{\alpha}\, f^{\alpha\mu} = \frac{1}{2}\, f^2\, u^{\mu} \tag{43}$$

Comparing these with (31) and (33) we get $c_1 - c_3 = 1$. Hence the only field equations remaining for the Einstein–aether theory are those given in (43). This result shows that the Einstein–aether and the Einstein theories are equivalent under the assumptions when the metric is the Gödel type and $c_1 - c_3 = 1$. In both cases the spacetime is curved due to the unit timelike vector field and the matter distribution is a charged dust due to the same unit timelike vector field.

Gödel type metrics solve also the special case of the aether theory (charged dust case) given in (16)–(18) but in our case the parameters $c_2$ and $c_4$ are not necessarily zero and $c_3 \neq -c_1$. We find $c_1 = \frac{1}{2}$ and $\lambda = \frac{f^2}{4}$. Here we take the zeroth component of the vector field as unity, $u_0 = 1$. When we relax this condition one needs to introduce a scalar field into the theory. Such a theory, called TeVeS, is given in [14]. We conjecture that Gödel type metrics with nonconstant $u_0$ form a class of exact solutions of TeVeS.

Here we have some remarks: (1) Gödel type metrics (26) we defined here and in the previous section differ from the metric redefinition in (10) because $h_{\alpha\beta}$ in Gödel type metrics is a degenerate matrix, its determinant is equal to zero. Hence the Gödel type metrics form a distinct class of exact solutions of the aether theory. (2) Although there are no closed timelike or closed null geodesics in Gödel type of spacetimes, they contain closed timelike and null worldlines [17]. It seems that violation of Lorentz invariance implements causality violations.

## 4 Conclusion

In [15] and [16] we have shown that Gödel type metrics arise in several low energy limits of string theory. In all these theories we have reduced the field equations to Maxwell type of equations in various dimensions. Several exact solutions with their properties were exhibited. Among the properties of these solutions we can mention the existence of closed timelike and closed null curves. In this work we carried these solutions, the Gödel type metrics, to the Einstein–aether theory. We proved that the Gödel type metrics reduce the complete field equations of the theory to three dimensional Maxwell equations corresponding to the unit timelike vector field where all parameters of the theory are left free except $c_1 - c_3 = 1$. We also showed that Gödel type metrics solve a special reduction of the aether theory [6,13,14].

The Gödel type metrics used in this work has $u_0 = 1$. If we relax this condition, these metrics constitute a class of exact solutions of several low energy limits of string theories in various dimensions with nonconstant dilaton field. We claim that the Gödel type metrics with nonconstant $u_0$ solve field equations of an aether theory, like TeVeS [14] or its modifications, with a dilaton field.

In aether theories, in addition to the spacetime metric a unit timelike vector field is considered which implies existence of preferred unit timelike direction at each point of spacetime, breaking the local Lorentz symmetry. An alternative to the timelike vector field we may consider a dynamical null vector. The action will be similar to the one given in (2) except the coefficient of the $\lambda$ term. We conjecture that Kerr–Schild metrics will form a class of exact solutions of such theories. All these issues will communicated later.

**Acknowledgments** This work is partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK) and Turkish Academy of Sciences (TUBA).

## References

1. Caroll, S.M., Harvey, J.A., Kostelecky, V.A., Lane, C.D., Okamoto, T.: Phys. Rev. Lett. **87**, 141601 (2001)
2. Kostelecky, A., Mewes, M.: Phys. Rev. Lett. **87**, 251304 (2001)
3. Lim, E.A.: Phys. Rev. D **71**, 063504 (2005)
4. Elling, C., Jacobson, T., Mattingly, D.: Einstein aether theory, based on a talk given by T. Jacobson at the Deserfest (2004)
5. Caroll, S.M., Lim, E.A.: Phys. Rev. D **70**, 123525 (2004)
6. Jacobson, T.: Einstein–aether gravity: theory and observational constraints. In: The Proceedings of the Meeting on CPT and Lorentz Symmetry (CPT 07), Bloomington, Indiana, 8–11 August 2007. arXiv.0711.3822
7. Jacobson, T., Mattingly, D.: Phys. Rev. D **64**, 024028 (2001)
8. Heinicke, C., Baekler, P., Hehl, F.W.: Phys. Rev. D **72**, 025012 (2005)
9. Foster, B.Z., Jacobson, T.: Phys. Rev. D **73**, 064015 (2006)
10. Tamaki, T., Miyamoto, U.: Generic features of Einstein-Aether black holes. Phys. Rev. D **77**, 024026 (2008)
11. Barbero G, J.F., Vilaseñor, E.J.S.: Phys. Rev. D **68**, 087501 (2003)
12. Foster, B.Z.: Phys. Rev. D **72**, 044017 (2005)
13. Zlosnik, T.G., Ferreira, P.G., Starkman, G.D.: Phys. Rev. D **74**, 044037 (2006)
14. Bekenstein, J.D.: Phys. Rev. D **70**, 083509. Erratum. Phys. Rev. D **71**, 069901 (2004)
15. Gürses, M., Karasu, A., Sarıoğlu, O.: Class. Quantum Grav. **22**, 1527–1543 (2005)
16. Gürses, M., Sarıoğlu, O.: Class. Quantum Grav. **22**, 4699 (2005)
17. Gleiser, R.J., Gürses, M., Karasu, A., Sarıoğlu, Ö.: Class. Quantum Grav. **23**, 2653 (2006)