How to Find the Absolute Maximum and the Absolute Minimum Values of a Continuous Function on a Closed and Bounded Region in the Plane

Theorem: Let $f(x, y)$ be a continuous function defined on a closed and bounded region R in the plane. Then f has an absolute maximum and an absolute minimum value on R.

This theorem is the analogue of the following theorem for 1-variable functions:

Theorem: Let $f(x)$ be a continuous function defined on a closed interval of finite length $[a, b]$. Then f has an absolute maximum and an absolute minimum value on $[a, b]$.

Let us recall how we find these values in the 1-variable case:

How to find the absolute maximum and the absolute minimum values of a continuous function $f(x)$ on a closed interval of finite length $[a, b]$:

1. Compute f'.
2. Find the critical points of f in (a, b).
3. Add the endpoints a and b of the interval $[a, b]$ to the list of points found in *step 2*.
4. Compute the value of f at each of the points in the list.
5. The largest value in *step 4* is the absolute maximum value of f on $[a, b]$ and the smallest value is the absolute minimum value.

There is a similar algorithm in the 2-variable case:

How to find the absolute maximum and the absolute minimum values of a continuous function $f(x, y)$ on a closed and bounded region R:

1. Compute f_x and f_y.
2. Find the critical points of f in the interior of R.
3. Find the critical points of the restriction of $f(x, y)$ to the boundary of R and add these to the list of points found in *step 2*.
4. Compute the value of f at each of the points in the list.
5. The largest value in *step 4* is the absolute maximum value of f on R and the smallest value is the absolute minimum value.

Example: Find the the absolute maximum and the absolute minimum values of $f(x,y) = x^3 - xy + y^2 - x$ on $R = \{(x,y) : x \geq 0, y \geq 0 \text{ and } x + y \leq 2\}$.

We apply the algorithm:

1. $f_x = 3x^2 - y - 1$ and $f_y = -x + 2y$.

2. We have to solve the equations $3x^2 - y - 1 = 0$ and $-x + 2y = 0$ simultaneously. From the second one we solve for x to get $x = 2y$ and substitute it in the first one to get $3 \cdot (2y)^2 - y - 1 = 0 \Longrightarrow 12y^2 - y - 1 = 0 \Longrightarrow y = 1/3$ or $y = -1/4$. Since $x = 2y$, these give the x-values $y = 1/3 \Longrightarrow x = 2/3$ and $y = -1/4 \Longrightarrow x = -1/2$. So we obtain the points $(x,y) = (2/3, 1/3)$ and $(x,y) = (-1/2, -1/4)$. Note that the second point is not in R. So we put only the first one, $(2/3, 1/3)$ in our list.

   This is important! If you include points that are not in R in your list, there is no guarantee that the algorithm will give the correct answer.

3. Now we come to the boundary. The boundary of R is the union of three sides of the triangle: *Side I*$= \{(x,y) : 0 \leq x \leq 2 \text{ and } y = 0\}$, *Side II*$= \{(x,y) : x = 0 \text{ and } 0 \leq y \leq 2\}$ and *Side III*$= \{(x,y) : x + y = 2 \text{ and } 0 \leq x \leq 2\}$. So we have essentially three 1-variable problems to solve:

   <u>*Side I*:</u> We parametrize this side in the obvious way: $x = t, y = 0$ for $0 \leq t \leq 2$. Then the restrictions of f to *Side I* is $f(t,0) = t^3 \cdot 0 - t \cdot 0 - 0^2 - t = t^3 - t$ for $0 \leq t \leq 2$. Now we find the list of points for this 1-variable optimization problem: $d/dt(f(t,0)) = 3t^2 - 1 = 0 \Longrightarrow t = 1/\sqrt{3}$ or $t = -1\sqrt{3}$. The second solution is not in the interval $[0,2]$, so we only take the first one $t = 1/\sqrt{3}$, and the endpoints of the interval $[0,2]$, $t = 0$ and $t = 2$. These correspond to the points $(x,y) = (1/\sqrt{3}, 0)$, $(x,y) = (0,0)$, and $(x,y) = (2,0)$.

   <u>*Side II*:</u> This is similar to the previous calculation: $f(0,t) = t^2$ for $0 \leq t \leq 2$. $d/dt(f(0,t)) = 2t = 0 \Longrightarrow t = 0$. This is one of the endpoints, the other one is $t = 2$, and they correspond to the points $(x,y) = (0,0)$ and $(x,y) = (0,2)$ in the plane.

   <u>*Side III*:</u> This time one possible parametrization is $x = t, y = 2 - t$ for $0 \leq t \leq 2$. This gives $f(t, 2-t) = t^3 + 2t^2 - 7t + 4$ for $0 \leq t \leq 2$. Then $d/dt(f(t, 2-t)) = 3t^2 + 4t - 7 = 0 \Longrightarrow t = 1$ or $t = -7/3$. Again the second solution is not in the interval we are looking at. The first one gives $t = 1 \Longrightarrow (x,y) = (t, 2-t) = (1,1)$. So our points, including the endpoints, are $(x,y) = (1,1)$, $(x,y) = (2,0)$ and $(x,y) = (0,2)$.

   To summarize, we found the points

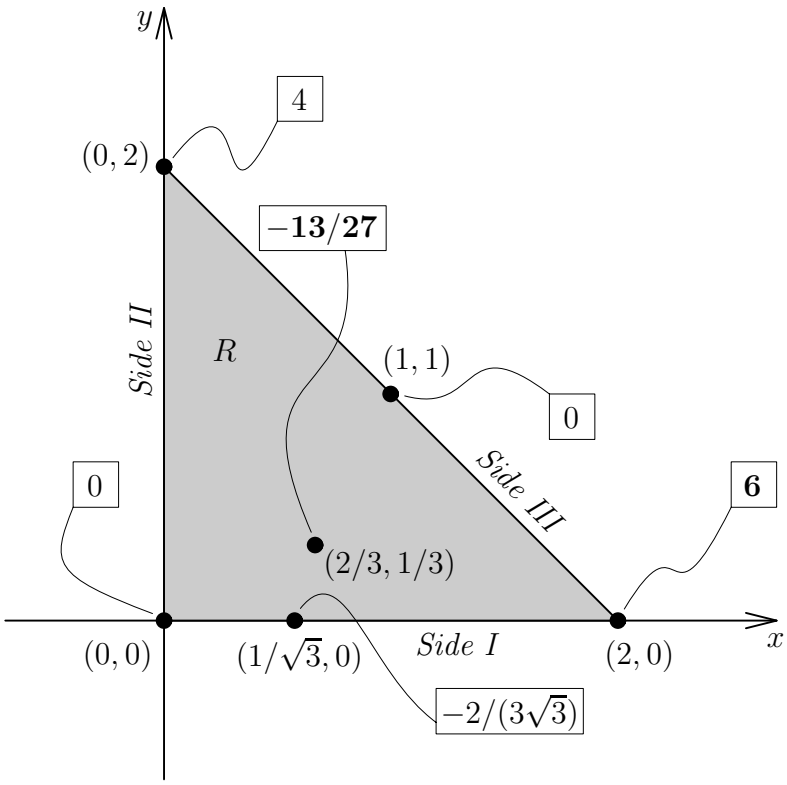
   $$(0,0), (2,0), (0,2), (1/\sqrt{3}, 0), (1,1)$$

   in this step. So now the complete list of points we are going to look at is

   $$(2/3, 1/3), (1/\sqrt{3}, 0), (0,0), (2,0), (0,2), (1,1) .$$

4. The values at these points are:

$$f(2/3, 1/3) = -\frac{13}{27} \qquad f(0,0) = 0 \qquad f(0,2) = 4$$

$$f(1/\sqrt{3}, 0) = -\frac{2}{3\sqrt{3}} \qquad f(2,0) = 6 \qquad f(1,1) = 0$$

5. Among these values, $-13/27$ is the smallest and 6 is the largest. Therefore, the absolute maximum value of f on R is 6, and it occurs at the point $(x, y) = (2, 0)$; and the absolute minimum value is $-13/27$ and it occurs at the point $(x, y) = (2/3, 1/3)$.