BİLKENT UNIVERSITY
Mathematics Department

Math 116 Intermediate Calculus III
Summer School 2006-2007

FIRST MIDTERM EXAM - Answer Key

4:00 pm - 6:00 pm (120 minutes) *June 17, 2007*

Surname : ....

Name : ....

Id. No. : ....

Section : ....

IMPORTANT

- This exam consists of 5 questions of equal weight.
- Each question is on a separate sheet. Please read the questions carefully and write your answers under the corresponding question. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get the full credit.
- Calculators are not allowed.

Please do not write anything below this line.

Q1	Q2	Q3	Q4	Q5	Total
20 pts.	20 pts.	20 pts.	20 pts.	20 pts.	100 pts.

Question 1 (10+10=20 points). Let

$$f(x,y) = \begin{cases} \dfrac{x^3\cos x - 4y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\ , \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Show that f is continuous at $(0,0)$.

Note that

$$\Big|\frac{x^3\cos x - 4y^3}{x^2+y^2} - 0\Big| \leq \frac{|x|x^2 + 4|y|y^2}{x^2+y^2} \leq |x| + 4|y| \leq 5\sqrt{x^2+y^2}$$

Thus, for any given $\epsilon > 0$, choose $\delta > 0$ as $\delta < \epsilon/5$, so that

$$\Big|\frac{x^3\cos x - 4y^3}{x^2+y^2} - 0\Big| < \epsilon \quad \text{when} \quad 0 < \sqrt{x^2+y^2} < \delta.$$

This proves that $f(x,y)$ is continuous at $(0,0)$ since $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(b) Compute $f_x(0,0)$ and $f_y(0,0)$ by using definition of partial derivatives.

$f(0,0) = 0,\quad f(h,0) = h\cos(h)$ and $f(0,k) = -4k$. Then

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{h\cos(h)}{h} = 1,$$

and

$$f_x(0,0) = \lim_{k\to 0}\frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0}\frac{-4k}{k} = -4.$$

Question 2 (10+10=20 points).

(a) Let $u(x,y) = xyf\Big(\frac{x+y}{xy}\Big)$ where $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function. Show that

$$x^2\frac{\partial u}{\partial x} - y^2\frac{\partial u}{\partial y} = G(x,y)u(x,y),$$

for some function $G(x,y)$ of two variables and find $G(x,y)$.

It just follows from the Chain rule:

$$\frac{\partial u}{\partial x} = yf\Big(\frac{x+y}{xy}\Big) + xy\frac{xy - y(x+y)}{x^2y^2}f'\Big(\frac{x+y}{xy}\Big)$$

Therefore,

$$\frac{\partial u}{\partial x} = yf\Big(\frac{x+y}{xy}\Big) - \frac{y}{x}f'\Big(\frac{x+y}{xy}\Big) \quad \text{and} \quad \frac{\partial u}{\partial y} = xf\Big(\frac{x+y}{xy}\Big) - \frac{x}{y}f'\Big(\frac{x+y}{xy}\Big)$$

Finally,

$$x^2\frac{\partial u}{\partial x} - y^2\frac{\partial u}{\partial y} = (x-y)xyf\Big(\frac{x+y}{xy}\Big) \quad \text{and hence,} \quad G(x,y) = x - y.$$

(b) Let $f(x,y,z) = \mathrm{e}^x\cos(yz)$. Find the directional derivative of f at the origin in the direction of $\mathbf{v} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.

The unit vector $\mathbf{u}$ in direction of $\mathbf{v}$ is given by

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$$

Since $\nabla f = \mathrm{e}^x\cos(yz)\mathbf{i} - z\mathrm{e}^x\sin(yz)\mathbf{j} - y\mathrm{e}^x\sin(yz)\mathbf{k}$. Then $\nabla f|_{(0,0,0)} = \mathbf{i}$ and

$$\Big(\mathrm{D}_{\mathbf{u}}f\Big)_{(0,0,0)} = \nabla f|_{(0,0,0)} \cdot \mathbf{u} = 3/7.$$

Question 3 (10+10=20 points). Let $f(x, y) = \frac{1}{2}x^2 + xy + \frac{1}{4}y^2 + 3\cos(x - 2) - 3y + 4$.

(a) Find the linearization $L(x, y)$ of $f(x, y)$ at the point $P(2, 1)$.

We have $f_x(x, y) = x + y - 3\sin(x - 2)$ and $f_y(x, y) = x + y/2 - 3$. So, $f_x(2, 1) = 3$ and $f_y(2, 1) = -1/2$ also $f(2, 1) = 33/4$.

Thus the linearization $L(x, y)$ of $f(x, y)$ at the point $P(2, 1)$ is given by

$$L(x, y) = \frac{33}{4} + 3(x - 2) - \frac{1}{2}(y - 1)$$

(b) Find an upper bound for the magnitude $|E(x, y)|$ of the error term $E(x, y)$ in the standard linear approximation $f(x, y) \approx L(x, y)$ over the rectangle $R : |x - 2| \leq 0.1$ and $|y - 1| \leq 0.3$

Now, we have $f_{xx}(x, y) = 1 - 3\cos(x - 2)$ $\;f_{xy}(x, y) = 1$ and $f_{yy}(x, y) = 1/2$. Hence, $|f_{xx}(x, y)| \leq 4$, $\;|f_{xy}(x, y)| \leq 4$ and $|f_{yy}(x, y)| \leq 4$.

Since,

$$|E(x, y)| \leq \frac{M}{2}\Big(|x - 2| + |y - 1|\Big)^2$$

We can take $M = 4$ and $|x - 2| \leq 0.1$ and $|y - 1| \leq 0.3$, then $|E(x, y)| \leq 0.32$.

Question 4 (20 points). The temperature function at all points in the disc $x^2+y^2 \leq 1$ is given by $T(x,y) = (x+y)\mathrm{e}^{-(x^2+y^2)}$. Find the highest and lowest temperatures at the points of the disc. Give the coordinates of points for the highest and lowest temperatures.

$$T_x(x,y) = (1-2x^2-2xy)\mathrm{e}^{-(x^2+y^2)} \text{ and } T_y(x,y) = (1-2y^2-2xy)\mathrm{e}^{-(x^2+y^2)}$$

The critical points are the solutions of the system below:

$$\begin{cases} 1-2x^2-2xy = 0 \\ 1-2y^2-2xy = 0 \end{cases}$$

So, $y = x$ or $y = -x$. The former implies that $1-2x^2 = 2x^2$, i.e, $x = \pm 1/2$ and the latter implies $1+2x^2 = 2x^2$ and consequently, there is no solution for the case of $y = -x$. Therefore, the critical points are $(1/2, 1/2)$ and $(-1/2, -1/2)$ and also note that $T(1/2, 1/2) = 1/\sqrt{\mathrm{e}}$ and $T(-1/2, -1/2) = -1/\sqrt{\mathrm{e}}$.

Use the method of Lagrange's Multipliers to find extremum values of the temperature function $T(x,y)$ on the boundary $x^2+y^2 = 1$, i.e, $\nabla T = \lambda \nabla g$, where $g(x,y) = x^2+y^2-1 = 0$. More precisely,

$$(1-2x^2-2xy)\mathrm{e}^{-(x^2+y^2)}\mathbf{i} + (1-2y^2-2xy)\mathrm{e}^{-(x^2+y^2)}\mathbf{j} = \lambda(2x\mathbf{i}+2y\mathbf{j}).$$

Since x and y are nonzero,

$$\frac{x}{y} = \frac{1-2x^2-2xy}{1-2y^2-2xy},$$

then $x = y$ and $x^2+y^2 = 1$, i.e, $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$ are the critical points of $T(x,y)$ on the boundary.
Notice that $T(1/\sqrt{2}, 1/\sqrt{2}) = \sqrt{2}/e$ and $T(-1/\sqrt{2}, -1/\sqrt{2}) = -\sqrt{2}/e$.

Therefore, the highest temperature $1/\sqrt{e}$ is taken at $(1/2, 1/2)$, and the lowest temperature $-1/\sqrt{e}$ is taken at $(-1/2, -1/2)$.

Question 5 (4+8+8=20 points). Let $f(x, y) = \mathrm{e}^{x^2-4y} + \ln(x - y^2)$ and $P(2, 1)$ be given.

(a) Find an equation of the level curve C of f that passes through $P(2, 1)$.

$f(2, 1) = \mathrm{e}^0 + \ln 1 = 1$, so the level curve C is

$$\mathrm{e}^{x^2-4y} + \ln(x - y^2) = 1.$$

(b) Find an equation of the tangent line to the level curve C in part **(a)** at the point $P(2, 1)$.

We have,

$$\nabla f = \Big(2x\mathrm{e}^{x^2-4y} + \frac{1}{x - y^2}\Big)\mathbf{i} - \Big(4\mathrm{e}^{x^2-4y} + \frac{2y}{x - y^2}\Big)\mathbf{j}$$

and

$$\nabla f|_{(2,1)} = 5\mathbf{i} - 6\mathbf{j}.$$

then the equation of the tangent line is

$$5(x - 2) - 6(y - 1) = 0.$$

or $5x - 6y = 4$

(c) Find an equation of the tangent plane of the surface $z = f(x, y)$ at the point $(2, 1, f(2, 1))$.

Define $g(x, y, z) = z - f(x, y)$, then

$$\nabla g = -\Big(2x\mathrm{e}^{x^2-4y} + \frac{1}{x - y^2}\Big)\mathbf{i} + \Big(4\mathrm{e}^{x^2-4y} + \frac{2y}{x - y^2}\Big)\mathbf{j} + \mathbf{k},$$

and hence $\nabla g|_{(2,1,f(2,1))} = -5\mathbf{i} + 6\mathbf{j} + \mathbf{k}$.

The tangent plane to the surface $z = f(x, y)$ at the point $(2, 1, f(2, 1))$ is given by

$$-5(x - 2) + 6(y - 1) + (z - 1) = 0.$$

or $5x - 6y - z = 3$