Math 101
Final Exam
May 23, 2011
15:30 - 17:30

Name : _____

ID# : _____

Department : _____

Section : _____

Instructor : _____

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully.
- **Show all your work in legibly written, well-organized mathematical sentences.**
- Calculators and dictionaries are not allowed.
- Simplify as far as possible unless otherwise stated.
- Please turn off your cellular phones before the exam starts.
- Extra sheets of paper, if needed, will be provided.

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Total
20	20	20	20	20	100

1 a) Find the limit

$$\lim_{x\to 1}\frac{\sin^2(x^3-1)}{(x-1)^2}$$

SOLUTION:

$$\lim_{x\to 1}\frac{\sin^2(x^3-1)}{(x-1)^2}=\lim_{x\to 1}\left(\frac{\sin(x^3-1)}{(x-1)}\right)^2=\left(\lim_{x\to 1}\frac{\sin(x^3-1)}{(x-1)}\right)^2$$

Using the L'Hospital's rule we get

$$\lim_{x\to 1}\frac{\sin(x^3-1)}{(x-1)}=\lim_{x\to 1}\frac{3x^2\,\cos(x^3-1)}{1}=3$$

Hence

$$\lim_{\mathbf{x\to 1}}\frac{\sin^{\mathbf{2}}(\mathbf{x^3-1})}{(\mathbf{x-1})^{\mathbf{2}}}=\mathbf{9}$$

or, by using the fact $x^3-1=(x-1)(x^2+x+1)$ then

$$\frac{\sin(x^3-1)}{x-1}=(x^2+x+1)\,\frac{\sin(x^3-1)}{x^3-1}$$

Hence

$$\lim_{x\to 1}\frac{\sin^2(x^3-1)}{(x-1)^2}=\lim_{x\to 1}\left((x^2+x+1)\,\frac{\sin(x^3-1)}{x^3-1}\right)^2=\lim_{x\to 1}(x^2+x+1)^2\,\lim_{x\to 1}\left(\frac{\sin(x^3-1)}{x^3-1}\right)^2=9$$

1 b) Let

$$f(x)=\begin{cases}x^2, & x\geq 0\\ -x^2, & x\leq 0.\end{cases}$$

Find $f'(x)$ and $f''(x)$ everywhere

SOLUTION:

i) $f'(x)=2x$ for $x>0$ and $f'(x)=-2x$ for $x<0$.
At the point $x=0$
$f'(0_-)=\lim_{h\to 0_-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0_-}\frac{-h^2}{h}=0$
$f'(0_+)=\lim_{h\to 0_+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0_+}\frac{h^2}{h}=0.$
Hence $f'(0)=0$

ii) $f''(x)=2$ for $x>0$ and $f''(x)=-2$ for $x<0$.
At the point $x=0$

$$f''(0_-) = \lim_{h\to 0_-} \frac{f'(h)-f'(0)}{h} = \lim_{h\to 0_-} \frac{-2h}{h} = -2$$
$$f''(0_+) = \lim_{h\to 0_+} \frac{f'(h)-f'(0)}{h} = \lim_{h\to 0_+} \frac{2h}{h} = 2.$$
Since $f''(0_-) \neq f''(0_+)$ then $f''(0)$ does not exist.

2 a) The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $P(1,1)$ intersects the curve at what other points? Draw the curve and its normal line at the point P.

SOLUTION: We have to find the equation of the normal line passing through the point $P(1,1)$. First we find the slope of the tangent line at $P(1,1)$. For this purpose we take the implicit derivative of the equation $x^2 + 2xy - 3y^2 = 0$.

$$2x + 2y + 2xy' - 6yy' = 0$$

or

$$y' = \frac{x+y}{3y-x}$$

Then at $P(1,1)$ we have $y'(1) = 1$, hence slope of the normal line is -1. Equation of the normal line

$$y - 1 = -(x-1) \quad \to y = -x + 2$$

Intersection points of the normal with the curve is found by inserting $y = -x + 2$ into $x^2 + 2xy - 3y^2 = 0$. We get

$$x^2 + 2x(-x+2) - 3(-x+2)^2 = 0 \quad \to x^2 - 4x + 3 = 0$$

Hence $x = 1, 3$. The normal line intersects the curve also at the the point $(3, -1)$

2 b) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12x - x^2$. What is the largest area that the rectangle can have and what are its dimensions?.

SOLUTION: Let the top of the rectangle be on the line $y = c$ where $c \leq 36$. The the left and right sides of the rectangle are respectively at the points $x_1 = 6 - \sqrt{36-c}$ and $x_2 = 6 + \sqrt{36-c}$. The length of the base is $x_2 - x_1$ and hight of the rectangle is c, hence the area A of the rectangle is

$$A = 2c\sqrt{36-c}, \qquad 0 \leq c \leq 36$$

Since A is zero at the end points the area takes its maximum value at an interior point

$$A'(c) = 2\sqrt{36-c} - \frac{c}{\sqrt{36-c}} = 0$$

which leads to $c = 24$. Then the maximal area is $A = 96\sqrt{3}$ and the dimensions of the rectangle are 24 and $2\sqrt{3}$.

3 a) Find the area of the surface generated by revolving about the x-axis the curve $y = 2\sqrt{x-1}$ with $1 \leq x \leq 2$

SOLUTION:

$$S = 2\pi \int y\, ds = 2\pi \int_1^2 2\sqrt{x-1}\sqrt{1+\frac{1}{x-1}}\, dx = 4\pi \int_1^2 \sqrt{x}\, dx = \frac{8\pi}{3}(x)^{3/2}|_1^2 = \frac{8\pi}{3}[2\sqrt{2}-1]$$

3 b) Find the length of the curve $y = \sqrt{1-x^2}$ with $\frac{1}{\sqrt{2}} \leq x \leq 1$.

SOLUTION:

$$L = \int ds = \int_{\frac{1}{\sqrt{2}}}^{2} \sqrt{1+(y')^2}\, dx = \int_{\frac{1}{\sqrt{2}}}^{1} \frac{dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1} x|_{\frac{1}{\sqrt{2}}}^{1} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

4 a) Find a formula for the nth partial sum of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

and use it to determine if the series converges or diverges. If the series converges, find its sum.

SOLUTION: For all $n \geq 2$

$$a_n = \frac{1}{n^2-1} = \frac{1}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

The sequence s_n of partial sums is given by

$$s_n = \sum_{k=2}^{n} \frac{1}{n^2-1} = \sum_{k=2}^{n} \frac{1}{2}\left[\frac{1}{k-1} - \frac{1}{k+1}\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$s_n = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right)$$

Since $\lim_{n\to\infty} s_n = \frac{3}{4}$ then the series converges and its sum $S = \frac{3}{4}$.

4 b) Evaluate

$$\int_0^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx$$

SOLUTION: This is an improper integral for two reasons. One of the integral limit is infinite and the integrand is not defined at the lower limit $x = 0$. The singularity of the integrand can be removed by a simple substitution. Let $u = \sqrt{x}$ then

$$\int_0^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = 2\int_0^{\infty} e^{-u}\, du = 2\lim_{a\to\infty} \int_0^{a} e^{-u}\, du$$

$$= 2\lim_{a\to\infty} \left(-e^{-u}|_0^a\right) = 2\lim_{a\to\infty} \left(-e^{-a}+1\right) = 2$$

5 a) Investigate the convergence of the series. Give reasons for your answer.

$$\sum_{n=1}^{\infty} \frac{e^{-\tan^{-1} n}}{1+n^2}$$

SOLUTION:
i) By the integral test: Let

$$f(x) = \frac{e^{-\tan^{-1} x}}{1+x^2}, \quad x \geq 1$$

One can show that $f(x)$ is a continuous, positive and decreasing function of x for all $x \geq 1$. Let us consider the integral

$$\int_1^{\infty} f(x)dx = \lim_{a\to\infty} \int_1^a \frac{e^{-\tan^{-1} x}}{1+x^2} dx = \lim_{a\to\infty} \int_{\pi/2}^{\tan^{-1} a} e^{-u} du$$

$$= \lim_{a\to\infty} \left(-e^{-u} \Big|_{\pi/2}^{\tan^{-1} a} \right) = e^{-\pi/4} - e^{-\pi/2}$$

Hence the series converges by **the integral test**.

ii) By the direct comparison test: Since $\tan^{-1} n < 1$ for all $n \geq 1$ we get

$$\frac{e^{-\tan^{-1} n}}{1+n^2} < \frac{1}{1+n^2}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges then the series $\sum_{n=1}^{\infty} \frac{e^{-\tan^{-1} n}}{1+n^2}$ converges by **the direct comparison test**.

5 b) For which $a \geq 0$ does the series

$$\sum_{n=1}^{\infty} \frac{a^{(n^2)}}{(n)!}$$

converge? Give reasons for your answer.

SOLUTION: By using the ratio test we get

$$\frac{a_{n+1}}{a_n} = \frac{a^{(n+1)^2}}{(n+1)!} \frac{n!}{a^{(n)^2}} = \frac{a^{(n+1)^2 - n^2}}{n+1} = \frac{a^{2n+1}}{n+1}$$

Hence

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a^{2n+1}}{n+1}$$

This limit exists and it is zero for $a \leq 1$. Hence the series converges for $a \leq 1$ by **the ratio test**.