Generating function of the Legendre function

$$g(x,t) = \sum_{n=0}^{\infty} t^n P_n(x) \qquad (1)$$

A recursion relation relation for $P_n(x)$

$$P'_{n+1}(x) - x P'_n(x) - (n+1) P_n = 0 \qquad (2)$$

using (1) and (2) it is easy to show that $g(x,t)$ satisfies the following first order p.d.e

$$\frac{1}{t} g_x - x\, g_x \cdot (t g)_t = 0 \qquad (3)$$

or

$$\left(\frac{1}{t} - x\right) g_x - t\, g_t = g \qquad (4)$$

using the Lagrange method the general solution of this equation is

$$g(x,t) = \frac{1}{t} f(\xi), \quad \xi = \frac{x}{t} - \frac{1}{2t^2} \qquad (5)$$

Here f is an arbitrary function to be determined by some initial conditions

From (1) we observe that

$$g(1,t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \qquad |t|<1 \qquad (6)$$

$$g(x,0) = 1 \qquad (7)$$

Hence using (6) we have

$$f\left(\frac{1}{t} - \frac{1}{2t^2}\right) = \frac{t}{1-t} \qquad (8)$$

one can solve f from (8) as

$$f(y) = \pm \frac{1}{\sqrt{1-2y}}$$

Hence

$$g(x,t) = \frac{1}{t} \frac{\pm 1}{\sqrt{1-2\xi}} = \frac{\pm 1}{\sqrt{t^2-2xt+1}} \qquad (9)$$

From (7) $g(x,0)=1$, hence we chose the solution with $+$ sign.

$$g(x,t) = \frac{1}{\sqrt{t^2-2xt+1}}$$

This is the GF function of the Legendre Polynomial.