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Kerr-Schild–Kundt metrics are universal

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Abstract

We define (non-Einsteinian) universal metrics as the metrics that solve the source-free covariant field equations of generic gravity theories. Here, extending the rather scarce family of universal metrics known in the literature, we show that the Kerr-Schild–Kundt class of metrics are universal. Besides being interesting on their own, these metrics can provide consistent backgrounds for quantum field theory at extremely high energies.

Keywords: universal metrics, Kerr-Schild–Kundt class, AdS waves, generic gravity theories

1. Introduction

The field equations of Einstein's gravity, even in vacuum $R_{\mu\nu} = 0$, are highly nonlinear, but still there is an impressive collection of exact solutions: some describing spacetimes outside compact sources, some describing nonlinear waves in curved or flat backgrounds, and some providing idealized cosmological spacetimes etc. According to the lore in effective field theories, the Einstein–Hilbert action will be modified, or one might say, quantum-corrected after heavy degrees of freedom in the microscopic theory are integrated out, with higher powers curvature and its derivatives at small distances/high energies. The ensuing theory at a given high energy scale could be a very complicated one with an action of the form

$$I = \int \mathrm{d}^D x \sqrt{-g} f(g, R, \nabla R, \ldots), \tag{1}$$

where f is a smooth function of its arguments, which are the metric g, the Riemann tensor denoted simply as R, the covariant derivative of Riemann tensor as ∇R, and the higher covariant derivatives of the Riemann tensor. Of course, it is quite possible that there are additionally nonminimally coupled fields such as scalar fields taking part in gravitation. But, in

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what follows we shall assume that this is not the case and gravity is simply described by the metric. This UV-corrected theory is much more complicated than Einstein's gravity, and so one might have *a priori* very little hope of finding exact solutions. Of course, what is even worse is that beyond the first few terms in perturbation theory, we do not really know the form of this modified theory at a given high energy scale. Hence, apparently, in the absence of the field equations, one may refrain from searching for solutions, but it turns out that the situation is not hopeless: there is an interesting line of research that started some time ago with the works [1–7] and culminated into a highly fertile research avenue. The idea is to find metrics, so called *universal metrics* [8], that solve all the metric-based field equations of quantum-corrected gravity, with slight modifications in the parameters that reflect the underlying theory. The notion of universal metrics, with refinements such as strongly and weakly universal were made in [8], we shall not go into that distinction here and we shall also not go into the distinction of critical versus non-critical metrics, where the former extremize an action while the latter solve a covariantly conserved field equation not necessarily coming from an action. These universal metrics, in addition to being valuable on their own, provide potentially consistent backgrounds for quantum field theory at extremely high energies where the backreaction or gravity of the quantum fields cannot be neglected. Universal Einsteinian (Ricci-flat or Einstein space) metrics were studied in the works [9, 10]. Non-Einsteinian universal metrics, such as the ones considered here, with or without cosmological constant are very rare.

From the above discussion, it should be clear that finding such universal metrics is a highly nontrivial task; hence, in the literature, there does not exist many examples save the ones we quoted above. But, recently, we have provided new examples of universal metrics: we have shown that the AdS-plane wave [13–15] (see also [16]) and the AdS-spherical wave [15, 17] metrics built on the (anti)-de Sitter [(A)dS] backgrounds solve generic gravity theories with an action of the form (1) or in general covariant field equations that satisfy a Bianchi identity [15, 18–20]. These previously found examples are in the form of the Kerr-Schild metrics[4] splitting as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu, \tag{2}$$

where $\bar{g}_{\mu\nu}$ represents the (A)dS spacetime and the λ vector satisfies the following four relations

$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}, \qquad \xi_\mu\lambda^\mu = 0, \qquad \lambda^\mu\partial_\mu V = 0. \tag{3}$$

Observe that a second vector ξ appears whose definition is given by the second relation, with the symmetrization convention defined as $2\xi_{(\mu}\lambda_{\nu)} \equiv \xi_\mu\lambda_\nu + \lambda_\mu\xi_\nu$. Note also that the λ vector is not a recurrent vector in general and hence the spacetime does not have the special holonomy group $\mathrm{Sim}(D-2)$ as was considered to be the case in [8]. With the second and third relations, the null λ vector becomes nonexpanding, shear-free, and nontwisting; making (2) a Kundt spacetime; therefore, we shall call this class of metrics as the Kerr-Schild–Kundt (KSK) class[5].

In this work, we prove that for any metric of the form (2) satisfying the conditions (3), the covariant field equations coming from the variation of (1) without any matter fields reduce to an equation linear in the traceless-Ricci tensor. This is the main purpose of this work. Once this reduction is achieved, one can have a further reduction in the field equations into a form that transparently shows that the solutions of Einstein's gravity and the quadratic curvature gravity in the KSK class are also solutions of generic gravity theories. The Einsteinian solutions are

[4] Higher dimensional Kerr-Schild spacetimes are extensively studied in [11, 12].

[5] The last condition in (3) is essential in showing the universality of the metrics although that property is not included in the definition of KSK metrics.

the members of the Type N universal spacetimes studied in [9]. In addition to these Einsteinian universal metrics, the solutions of the quadratic curvature gravity in the KSK class also solve the metric-based source-free field equations of any generic gravity theory, that is these metrics are *non-Einsteinian* universal metrics. As we stated above, the AdS-plane wave and the AdS-spherical wave metrics belong to the non-Einsteinian KSK family of metrics as being solutions of the quadratic curvature gravity theories. In addition, rather recently, we proposed a solution generation technique [20] to construct non-Einsteinian universal metrics and we found a new member of this class which is the dS-hyperbolic wave metric [21].

For metrics the form (2) satisfying the conditions (3), the vacuum field equations of the generic gravity theory with the action (1) can be written as

$$E_{\mu\nu} \equiv e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0, \tag{4}$$

as an immediate consequence of theorem 1 to be proven in section 3. Here, $S_{\mu\nu}$ is the traceless-Ricci tensor, and $\Box$ is the d'Alembert operator with respect to the metric $g_{\mu\nu}$. The derivative order of the generic theory is $2N+2$ such that $N=0$ is Einstein's gravity (or the Einstein–Gauss-Bonnet theory) and $N=1$ is the quadratic curvature gravity (or $f(g,R)$ where R represents the Riemann tensor). The field equations split into a single trace part and a higher derivative nonlinear wave equation for the traceless part. Taking the trace of this equation yields a scalar equation

$$e = 0, \tag{5}$$

which determines the effective cosmological constant in terms of the parameters of the theory, such as the bare cosmological constant and the dimensionful parameters that appear in front of the curvature invariants. On the other hand, the traceless part is a nontrivial nonlinear equation

$$\sum_{n=0}^{N} a_n \Box^n S_{\mu\nu} = 0. \tag{6}$$

This reduction is highly impressive, but in this form, the above equation cannot be solved save for some trivial cases. Hence, a further reduction is needed. It was shown in [19] that this is possible as

$$\Box^n S_{\mu\nu} = (-1)^n \lambda_\mu \lambda_\nu \left(\mathcal{O} + \frac{2}{\ell^2}\right)^n \mathcal{O} V. \tag{7}$$

Here, the operator $\mathcal{O}$ is defined as

$$\mathcal{O} \equiv \Box + 2\xi^\mu \partial_\mu + \frac{1}{2}\xi^\mu \xi_\mu - \frac{2(D-2)}{\ell^2} = \bar{\Box} + 2\xi^\mu \partial_\mu + \frac{1}{2}\xi^\mu \xi_\mu - \frac{2(D-2)}{\ell^2}, \tag{8}$$

where $\bar{\Box}$ is the d'Alembert operator with respect to the background metrics $\bar{g}_{\mu\nu}$ and $S_{\mu\nu} = -\lambda_\mu \lambda_\nu \mathcal{O} V$. This result given in (7) is valid for the KSK class with any ξ_μ satisfying (3), and using this, (6) reduces to a linear equation

$$\lambda_\mu \lambda_\nu \sum_{n=0}^{N} a_n (-1)^n \left(\mathcal{O} + \frac{2}{\ell^2}\right)^n \mathcal{O} V = 0. \tag{9}$$

For $N \geqslant 1$, this equation can be factorized as

$$\prod_{n=0}^{N} (\mathcal{O} + b_n)\, \mathcal{O} V = 0, \tag{10}$$

where b_n is related to a_ns and so to the parameters of the theory; albeit, in general, in a complicated implicit way. If all b_ns are distinct and none is zero, the most general solution of (10) is in the form

$$V = V_{\rm E} + V_{\rm I} + V_2 + \cdots + V_N, \tag{11}$$

where $V_{\rm E}$ is the Einsteinian solution satisfying

$$\mathcal{O}V_{\rm E} = 0, \tag{12}$$

and V_n is the solution of the quadratic curvature gravity satisfying

$$(\mathcal{O} + b_n)\, V_n = 0, \tag{13}$$

for all $n = 1, 2, \cdots, N$. For example, when $N = 1$, $V = V_{\rm E} + V_{\rm I}$ represents the quadratic curvature gravity solutions which also solve the generic theory. On the other hand, if some b_ns coincide or vanish, then genuinely fourth or higher power operators, such as $(\mathcal{O} + b_n)^2$, arise with Log-type solutions having asymptotically non-AdS behavior which exist in the so-called critical theories. Since $\mathcal{O}$ given in (8) is an operator which solely depends on the background metrics (flat, AdS, or dS), the solutions of (12) and (13) for $V_{\rm E}$ and V_n can easily be obtained by using some known techniques such as the method of separation of variables or the method of Green's function. As we have studied such issues in other works such as [13, 15, 17], here we shall not consider particular cases but give a detailed proof of how KSK metrics are universal provided that the equations (12) and (13) are solved for the functions $V_{\rm E}$ and V_n. In the rest of the paper, we call the KSK metrics where the metric function V solves (12)–(13) as universal.

The layout of the paper is as follows: In section 2, we give the curvature properties of the KSK metrics as well as the relations satisfied by the two special vectors λ and ξ that are important in description of these spacetimes. Section 3 constitutes the bulk of the paper where we show that the KSK metrics are universal. In the appendix, we give an alternative proof by mathematical induction. As our claim is strong, we were compelled to give two proofs which can be read independently. The one in the bulk of the paper is shorter but the one in the Appendix comes with various examples that will help the reader appreciate the construction.

2. Curvature tensors and properties of Kerr-Schild–Kundt class

In what follows, D will denote the number of dimensions of the spacetime. The properties of the KSK type metrics were previously discussed in [17, 19]. Here, we shall briefly recapitulate some of these which will be crucial in the proof and we shall also give some additional constructions in this section. The scalar curvature of KSK metrics is constant and normalized[6] as $R = -D(D-1)/\ell^2$ and the traceless-Ricci tensor, $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{D} g_{\mu\nu}$, can be shown to satisfy

$$S_{\mu\nu} = \rho \lambda_\mu \lambda_\nu, \tag{14}$$

where course λ_μ is the vector appearing in metric (2) and the new object ρ is given in terms of an operator acting on the profile function V as

$$\rho = -\mathcal{O}V = -\left(\Box + 2\xi^\mu \partial_\mu + \frac{1}{2}\xi^\mu \xi_\mu - \frac{2(D-2)}{\ell^2}\right) V. \tag{15}$$

[6] Here, the relation between the effective cosmological constant Λ and the AdS radius ℓ is given as

$$-\frac{1}{\ell^2} \equiv \frac{2\Lambda}{(D-1)(D-2)}.$$

This expression is not difficult to obtain, but a more involved computation gives the Weyl tensor as[7]

$$C_{\mu\alpha\nu\beta} = 4\lambda_{[\mu}\,\Omega_{\alpha][\beta}\,\lambda_{\nu]}, \tag{16}$$

where the symmetric two-tensor $\Omega_{\alpha\beta}$ is given as

$$\Omega_{\alpha\beta} \equiv -\left[\nabla_\alpha\partial_\beta + \xi_{(\alpha}\partial_{\beta)} + \frac{1}{2}\xi_\alpha\xi_\beta - \frac{1}{D-2}g_{\alpha\beta}\left(\mathcal{O} + \frac{2(D-2)}{\ell^2}\right)\right]V, \tag{17}$$

Its contraction with the λ vector and its trace read

$$\lambda^\alpha\,\Omega_{\alpha\beta} = \frac{1}{2}\lambda_\beta\,\Omega^\alpha_\alpha, \qquad \Omega^\alpha_\alpha = \xi^\alpha\partial_\alpha V - \frac{2}{D-2}\rho + \frac{4}{\ell^2}V,$$

which make it clear that the Weyl tensor satisfies $\lambda^\mu C_{\mu\alpha\nu\beta} = 0$. Observe that just like the metric function V, due to the Bianchi identity and the constancy of the scalar curvature, one has $\nabla^\mu S_{\mu\nu} = 0$ yielding

$$\lambda^\mu\nabla_\mu\rho = 0, \tag{18}$$

which also follows from an explicit calculation using the definition (15) and $\lambda^\mu\nabla_\mu V = 0$.

Let us now calculate the Riemann tensor: using the decomposition

$$R_{\mu\alpha\nu\beta} = C_{\mu\alpha\nu\beta} + \frac{2}{D-2}(g_{\mu[\nu}S_{\beta]\alpha} - g_{\alpha[\nu}S_{\beta]\mu}) + \frac{2R}{D(D-1)}g_{\mu[\nu}g_{\beta]\alpha}, \tag{19}$$

one arrives at a compact form for the KSK metrics

$$R_{\mu\alpha\nu\beta} = 4\lambda_{[\mu}\Theta_{\alpha][\beta}\lambda_{\nu]} + \frac{2R}{D(D-1)}g_{\mu[\nu}g_{\beta]\alpha}, \tag{20}$$

where $\Theta_{\alpha\beta}$ is defined in terms ρ and $\Omega_{\alpha\beta}$ as

$$\Theta_{\alpha\beta} \equiv \Omega_{\alpha\beta} + \frac{1}{D-2}\rho g_{\alpha\beta} = -\left(\nabla_\alpha\partial_\beta + \xi_{(\alpha}\partial_{\beta)} + \frac{1}{2}\xi_\alpha\xi_\beta - \frac{2}{\ell^2}g_{\alpha\beta}\right)V. \tag{21}$$

We shall make use of this form of the Riemann tensor in the next section. The trace and λ^α contraction of the two-tensor $\Theta_{\alpha\beta}$ are

$$\Theta^\alpha_\alpha = \rho + \xi^\alpha\partial_\alpha V + \frac{4V}{\ell^2}, \qquad \lambda^\alpha\Theta_{\alpha\beta} = \frac{1}{2}\lambda_\beta(\Theta^\alpha_\alpha - \rho). \tag{22}$$

All of these expressions are exact even though the metric function V appears linearly, which shows the remarkable property of the Kerr-Schild metrics in addition to the properties we have listed, defining the KSK class.

Finally, for the KSK metrics, we need the following identities: once-contracted Bianchi identity

$$\nabla^\nu R_{\mu\alpha\nu\beta} = \nabla_\mu R_{\alpha\beta} - \nabla_\alpha R_{\mu\beta}, \tag{23}$$

for constant R yields

$$\nabla^\nu R_{\mu\alpha\nu\beta} = \nabla_\mu S_{\alpha\beta} - \nabla_\alpha S_{\mu\beta}, \tag{24}$$

which then leads to the double-divergence of the Riemann tensor

[7] The anti-symmetrization with the square brackets is weighted with 1/2.

$$\nabla^{\mu}\nabla^{\nu}R_{\mu\alpha\nu\beta} = \left(\Box - \frac{R}{D-1}\right)S_{\alpha\beta}. \tag{25}$$

In obtaining this identity, we made use of $\nabla^{\mu}\nabla_{\sigma}S_{\mu\nu} = \frac{R}{D-1}S_{\sigma\nu}$ which follows from

$$\nabla^{\mu}\nabla_{\sigma}S_{\mu\nu} = [\nabla_{\mu}, \nabla_{\sigma}]\, S^{\mu}_{\nu} = R_{\sigma\alpha}S^{\alpha}_{\nu} + R^{\alpha}_{\mu\sigma\nu}S^{\mu}_{\alpha}, \tag{26}$$

after using the contractions $R_{\sigma\alpha}S^{\alpha}_{\nu} = \frac{R}{D}S_{\sigma\nu}$ and $R_{\mu\alpha\nu\beta}S^{\mu\beta} = \frac{R}{D(D-1)}S_{\nu\alpha}$.

The ξ vector that does not appear in the metric but appears in the definition of the KSK class will play an important role in the proof below; therefore, let us work out some of the identities that it satisfies:

$$\lambda^{\nu}\nabla_{\mu}\xi_{\nu} = -\frac{1}{2}\lambda_{\mu}\xi^{\nu}\xi_{\nu}, \tag{27}$$

and its divergence is

$$\nabla_{\mu}\xi^{\mu} = -\frac{1}{4}\xi^{\mu}\xi_{\mu} + \frac{2D-3}{D(D-1)}R. \tag{28}$$

We also have

$$\lambda^{\mu}\nabla_{\mu}\xi_{\alpha} = -\lambda_{\alpha}\left(\frac{1}{4}\xi^{\mu}\xi_{\mu} - \frac{1}{D(D-1)}R\right). \tag{29}$$

The first equality is simply due to $\lambda^{\nu}\xi_{\nu} = 0$. To obtain the second[8] and the third identities, let us note that we have $\left[\nabla_{\mu}, \nabla_{\nu}\right]\lambda_{\beta} = R_{\mu\nu\beta}{}^{\rho}\lambda_{\rho}$, whose right-hand side reduces to

$$R_{\mu\nu\beta}{}^{\rho}\lambda_{\rho} = \frac{R}{D(D-1)}\left(g_{\mu\beta}\lambda_{\nu} - \lambda_{\mu}g_{\nu\beta}\right), \tag{30}$$

after using (19) and the fact that the KSK spacetime is type-N Weyl (16) and type-N traceless-Ricci (14) [22, 23]. On the other hand, the left-hand side, $[\nabla_{\mu}, \nabla_{\nu}]\,\lambda_{\beta}$, takes the form

$$[\nabla_{\mu}, \nabla_{\nu}]\,\lambda_{\beta} = \lambda_{[\nu}\nabla_{\mu]}\xi_{\beta} - \lambda_{\beta}\nabla_{[\nu}\xi_{\mu]} - \frac{1}{2}\xi_{\beta}\lambda_{[\nu}\xi_{\mu]}, \tag{31}$$

after using $\nabla_{\mu}\lambda_{\nu} = \xi_{(\mu}\lambda_{\nu)}$ recursively. Overall, one has

$$2\lambda_{[\nu}\nabla_{\mu]}\xi_{\beta} - 2\lambda_{\beta}\nabla_{[\nu}\xi_{\mu]} - \xi_{\beta}\lambda_{[\nu}\xi_{\mu]} = \frac{2R}{D(D-1)}(g_{\mu\beta}\lambda_{\nu} - \lambda_{\mu}g_{\nu\beta}), \tag{32}$$

which can be used to find $\nabla_{\mu}\xi^{\mu}$ and $\lambda^{\mu}\nabla_{\mu}\xi_{\alpha}$ after performing the $g^{\mu\beta}$ and λ^{μ} contractions yielding

$$\lambda^{\mu}\nabla_{\mu}\xi_{\nu} = -\lambda_{\nu}\left(\nabla_{\mu}\xi^{\mu} + \frac{1}{2}\xi^{\mu}\xi_{\mu} - \frac{2R}{D}\right), \tag{33}$$

[8] A variation of (28) appeared in the appendix B of [17] such that it involves the covariant derivative with respect to the Christoffel connection of AdS, that is $\bar{\nabla}_{\mu}$. Thus, another way to obtain (28) is to show the equivalence $\bar{\nabla}_{\mu}\xi^{\mu} = \nabla_{\mu}\xi^{\mu}$. This result immediately follows from the fact that the Christoffel connection of the AdS spacetime is related to the Christoffel connection of the full metric as (see, for example, appendix B of [17])

$$\Omega^{\mu}{}_{\alpha\beta} \equiv \Gamma^{\mu}_{\alpha\beta} - \bar{\Gamma}^{\mu}_{\alpha\beta} = \bar{\nabla}_{\alpha}\left(V\lambda^{\mu}\lambda_{\beta}\right) + \bar{\nabla}_{\beta}(V\lambda^{\mu}\lambda_{\alpha}) - \bar{\nabla}^{\mu}\left(V\lambda_{\alpha}\lambda_{\beta}\right),$$

and using the fact that $\Omega^{\mu}{}_{\mu\beta} = 0$, one has $\bar{\Gamma}^{\mu}_{\mu\beta} = \Gamma^{\mu}_{\mu\beta}$.

$$\lambda_\nu\lambda^\mu\nabla_\mu\xi_\beta+\lambda_\beta\lambda^\mu\nabla_\mu\xi_\nu=-\lambda_\beta\lambda_\nu\left(\frac{1}{2}\xi^\mu\xi_\mu-\frac{2R}{D(D-1)}\right), \tag{34}$$

respectively, with the use of (27). Then, using (33) in (34) yields the equation (28) and making use of that equation in (33) yields (29).

The identities (27) and (29) play a crucial role in the proof below, because they represent the fact that all possible contractions of $\nabla_\mu\xi_\nu$ with a λ vector yields a free-index λ vector and a reduction in the order of the derivative on the ξ vector by one.

The vector $\partial_\mu V$ also satisfies similar properties like ξ_μ: for both of these vectors, contraction with λ^μ is zero and contractions of $\nabla_\mu\partial_\nu V$ with a λ vector satisfy

$$\lambda^\mu\nabla_\mu\partial_\nu V=\lambda^\mu\nabla_\nu\partial_\mu V=-\frac{1}{2}\lambda_\nu\xi^\mu\partial_\mu V, \tag{35}$$

where again a free-index λ vector appears and the order of the derivative on $\partial_\mu V$ reduces by one. With this background information, we are now ready to state and give the proof of the theorem in the next session.

3. Universality Of KSK metrics

Here, we are going to prove the following theorem:

Theorem 1. *For the Kerr-Schild metrics*

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+2V\lambda_\mu\lambda_\nu,$$

with the properties

$$\lambda^\mu\lambda_\mu=0,\qquad \nabla_\mu\lambda_\nu=\xi_{(\mu}\lambda_{\nu)},\qquad \xi_\mu\lambda^\mu=0,\qquad \lambda^\mu\partial_\mu V=0,$$

where $\bar{g}_{\mu\nu}$ is the metric of a space of constant curvature (AdS or dS), any second rank symmetric tensor constructed from the Riemann tensor and its covariant derivatives can be written as a linear combination of $g_{\mu\nu}$, $S_{\mu\nu}$, and higher derivatives of $S_{\mu\nu}$ in the form $\Box^n S_{\mu\nu}$ where $\Box$ represents the d'Alembertian with respect to $g_{\mu\nu}$, that is

$$E_{\mu\nu}=eg_{\mu\nu}+\sum_{n=0}^{N}a_n\Box^n S_{\mu\nu}.$$

Proof. The proof of this theorem relies on the observation that any contraction of the λ vector with any tensor composed of V and its covariant derivatives, ξ and its covariant derivatives always yields a free-index λ vector in each term in the resulting expression. Thus, in constructing two-tensors out of the contractions of any number of Riemann tensor and its derivatives, one must keep track of the number of λ vectors.

Let us consider a generic two-tensor which is constructed by any number of Riemann tensors and its covariant derivatives. We represent this two-tensor symbolically as

$$E_{\mu\nu}\equiv[R^{n_0}(\nabla^{n_1}R)(\nabla^{n_2}R)\dots(\nabla^{n_m}R)]_{\mu\nu}\,, \tag{36}$$

where R represents the Riemann tensor, the superscripts represent the number of terms involved such as n_0 represents the number of Riemann tensors without covariant derivatives, and $n_1\leqslant n_2\leqslant\cdots\leqslant n_m$ is assumed without loss of generality. In the notation of this section, the

Riemann tensor given in (20) can be simply given as $R = \lambda^2\Theta + g^2$. In the above expression, we omitted the metric tensors among the terms, and in principle, any contraction pattern is possible. The presence of these metric tensors does not alter any of our discussions below. It is obvious that to have a two-tensor, the sum $\sum_{i=1}^{m} n_i$ should be even. Considering the metric compatibility condition and using the form of the Riemann tensor in (20), $E_{\mu\nu}$ reduces to (say a new tensor $\mathcal{E}_{\mu\nu}$)

$$\mathcal{E}_{\mu\nu} \equiv [\lambda^{2n_0}\Theta^{n_0}(\nabla^{n_1}[\lambda^2\Theta])(\nabla^{n_2}[\lambda^2\Theta])\dots(\nabla^{n_m}[\lambda^2\Theta])]_{\mu\nu}\,, \tag{37}$$

where we omitted the metrics coming out of the Riemann tensors R^{n_0}, since considering them just yields a sum of two-tensor forms updated with $\lambda^{2n_r}\Theta^{n_r}$ instead of $\lambda^{2n_0}\Theta^{n_0}$ where $n_r < n_0$ always, so these terms are genuinely covered in $\mathcal{E}_{\mu\nu}$.

Now, let us consider the tensorial structures appearing in $\mathcal{E}_{\mu\nu}$. First, note that Θ defined in (21) is composed of V and its first and second order derivatives in addition to the ξ vector. Secondly, let us consider the highest order derivative term $(\nabla^{n_m}[\lambda^2\Theta])$ which is a $(0, n_m + 4)$ rank tensor. Note that with each application of the covariant derivative on λ, one can use $\nabla_\mu\lambda_\nu = \xi_{(\mu}\lambda_{\nu)}$; and therefore, $(\nabla^{n_m}[\lambda^2\Theta])$ represents a sum of $(0, n_m + 4)$ rank tensors that are built with V and its up to $(n_m + 2)$th-order derivatives in addition to the ξ vector and its n_mth-order derivatives. Therefore, the $\left(0, s \equiv 4n_0 + 4m + \sum_{i=1}^{m} n_i\right)$ rank tensor,

$$\mathcal{E}_{\mu_1\dots\mu_s} \equiv [\lambda^{2n_0}\Theta^{n_0}(\nabla^{n_1}[\lambda^2\Theta])(\nabla^{n_2}[\lambda^2\Theta])\dots(\nabla^{n_m}[\lambda^2\Theta])]\,, \tag{38}$$

represents a sum of $(0, s)$ rank tensors which are built with $2(n_0 + m)$ number of λ vectors and the remaining $(0, s - 2n_0 - 2m)$ rank tensorial parts are built with V and its up to $(n_m + 2)$th-order derivatives in addition to the ξ vector and its n_mth-order derivatives.

After discussing the tensorial structure of $\mathcal{E}_{\mu_1\dots\mu_s}$, now let us analyze the nature of the $(s/2 - 1)$ number of contractions with the inverse metric yielding $\mathcal{E}_{\mu\nu}$. First, note that the contractions of the λ^μ vector with λ_μ, ξ_μ, and $\partial_\mu V$ yield zero. Secondly, the contractions of the λ^μ vector with the first order derivatives of ξ_μ and $\partial_\mu V$ yield (27) and (29), and (35), respectively. In these contractions, the important points to observe are:

- the number of the λ vectors is preserved since a free-index λ always appears in the results,
- contraction with the λ vector removes the first order derivatives acting on ξ_μ and $\partial_\mu V$.

Now, let us analyze the λ^μ contraction of the terms involving higher order covariant derivatives acting on ξ_μ and $\partial_\mu V$. Note that to arrive at the stated proof, instead of explicit formulae, the tensorial structure of the expressions after the λ^μ contractions is important. Since the λ^μ contractions of both ξ_μ and $\partial_\mu V$ yield the same structure, we worked with ξ_μ for definiteness; however, the conclusions we obtained are also valid in the $\partial_\mu V$ case. Thus, let us consider the $(0, r + 1)$ rank tensor in the form

$$\nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu_r}\xi_{\mu_{r+1}}. \tag{39}$$

The λ^μ contraction can be through one of the covariant derivatives as

$$\lambda^\mu\nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu}\dots\nabla_{\mu_{r-1}}\xi_{\mu_r}, \tag{40}$$

or through the ξ vector as

$$\lambda^{\mu}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_r}\xi_{\mu}. \tag{41}$$

For these two contraction patterns, the tensorial structure of the final results are sums of the $(0, r)$ rank tensors satisfying the properties;

- each term involves a free-index λ vector,
- for all the terms, the highest order of derivative acting on ξ will be $r-1$ or less.

To show these properties, we need to use the basic identities (27) and (29), and to make such a use, first, one needs to change the orders of the derivatives in (40) such that one has

$$\lambda^{\mu}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}, \tag{42}$$

by using the Ricci identity[9] producing Riemann tensors for each change of order. After making all the change of orders and applying simply the product rule for the covariant derivatives, one arrives at

$$\lambda^{\mu}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu}\ldots\nabla_{\mu_{r-1}}\xi_{\mu_r} = \lambda^{\mu}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r} + \sum_{p}\lambda^{\mu}(\nabla^{p}\boldsymbol{R}_{\mu})(\nabla^{r-p-2}\xi), \tag{43}$$

where in the last sum, the $\lambda^{\mu}(\nabla^{p}\boldsymbol{R}_{\mu})$ term represents p number of covariant derivatives acting on the Riemann tensor and one index of the Riemann tensor should be contracted with λ^{μ}. Here, p can have various values depending on the position of the contracted covariant derivative in (40) and it can be as small as 0 and as large as $(r-2)$. Once we consider the Riemann tensor R symbolically as $\lambda^2\Theta$, then

$$\lambda^{\mu}(\nabla^{p}\boldsymbol{R}_{\mu}) = \lambda^{\mu}(\nabla^{p}[\lambda^{2}\Theta]_{\mu}), \tag{44}$$

represents a sum of terms involving two free-index λ vectors and the remaining $(0, p+1)$-rank tensor structure is built with the $\xi, \partial V$ vectors, and their covariant derivatives. In each term in this summation, one higher order covariant derivative term involving ξ or ∂V must have a λ^{μ} contraction. The derivative order of this λ^{μ} contracted term is at most $(r-1)$ for the ∂V vector and $(r-2)$ for the ξ vector. This is because Θ involves the first derivative of the ∂V vector and just the ξ vector itself, and p can take the maximum value of $(r-2)$. To summarize, for the last sum in (43), the properties of the tensorial structure of each term is:

- there are three λ vectors one of which is in the contracted form and the others are free,
- the total number of derivatives in these terms is at most $(r-1)$ for ∂V and $(r-2)$ for ξ, so the order of the derivative is reduced by 1.

So, for these terms, we achieved to show the aimed two properties.

Now, let us focus on the first term in (43) and (41). For these terms, we need to change the order of the covariant derivatives and the λ^{μ} vector such that in the end we obtain

$$\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}(\lambda^{\mu}\nabla_{\mu}\xi_{\mu_r}), \tag{45}$$

[9] Here, with Ricci identity, we mean

$$\left[\nabla_{\mu},\nabla_{\nu}\right]T_{\alpha\beta\ldots\gamma} = R_{\mu\nu\alpha}{}^{\lambda}T_{\lambda\beta\ldots\gamma} + \cdots + R_{\mu\nu\gamma}{}^{\lambda}T_{\alpha\beta\ldots\lambda}.$$

$$\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}(\lambda^{\mu}\nabla_{\mu_r}\xi_{\mu}), \tag{46}$$

respectively, and we can apply the identities (29) and (27) in these terms. To show how we carry out this simple change of orders, we consider the first term in (43) and the same steps apply for (41). In commuting the λ^{μ} vector and the covariant derivatives, we simply have

$$\lambda^{\mu}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r} = \nabla_{\mu_1}(\lambda^{\mu}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}) - (\nabla_{\mu_1}\lambda^{\mu})\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}, \tag{47}$$

where in the second term on the right-hand side, one can apply the defining property of the ξ vector $\nabla_{\mu}\lambda_{\nu} = \xi_{(\mu}\lambda_{\nu)}$ which reduces the derivative order and introduces a free-index λ vector. Then, one has

$$\begin{aligned}\lambda^{\mu}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r} = {}& \nabla_{\mu_1}\Big(\lambda^{\mu}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}\Big)\\ & -\frac{1}{2}\xi_{\mu_1}\lambda^{\mu}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}\\ & -\frac{1}{2}\lambda_{\mu_1}\xi^{\mu}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r},\end{aligned} \tag{48}$$

where for the last term, we achieved our aim that

- a free-index λ vector is introduced,
- the derivative order on ξ_{μ_r} is reduced by one.

On the other hand, the second term in (48) still involves a λ^{μ} contraction; but this time, the order of the derivative acting on ξ_{μ_r} is $(r-1)$. For this term, one needs to repeat this ongoing process for the generic rth-derivative term. For the next step of the change of orders, we consider the first term on the right-hand side (48) and change the order of λ^{μ} and ∇_{μ_2} as

$$\begin{aligned}\lambda^{\mu}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r} = {}& \nabla_{\mu_1}\nabla_{\mu_2}\Big(\lambda^{\mu}\nabla_{\mu_3}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}\Big)\\ & -\Big(\nabla_{\mu_1}\nabla_{\mu_2}\lambda^{\mu}\Big)\Big(\nabla_{\mu_3}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}\Big)\\ & -\Big(\nabla_{\mu_2}\lambda^{\mu}\Big)\nabla_{\mu_1}\nabla_{\mu_3}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}\\ & -\frac{1}{2}\xi_{\mu_1}\lambda^{\mu}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}\\ & -\frac{1}{2}\lambda_{\mu_1}\xi^{\mu}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\nabla_{\mu}\xi_{\mu_r}.\end{aligned} \tag{49}$$

Here, again using $\nabla_{\mu}\lambda_{\nu} = \xi_{(\mu}\lambda_{\nu)}$ in the second and third terms yield either λ^{μ} contracted terms having less number of derivatives than r acting on ξ or terms involving a free-index λ vector. Again for the terms involving the λ^{μ} contraction this ongoing procedure can be repeated. Thus, one can continue changing the order of the λ^{μ} vector and the covariant derivatives in the first term until one arrives at

$$\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}(\lambda^{\mu}\nabla_{\mu}\xi_{\mu_r}), \tag{50}$$

which reduces to

$$\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{r-1}}\left[-\lambda_{\mu_r}\left(\frac{1}{4}\xi^{\mu}\xi_{\mu}-\frac{1}{D(D-1)}R\right)\right], \tag{51}$$

after making use of (29). This term after the use of $\nabla_\mu\lambda_\nu=\xi_{(\mu}\lambda_{\nu)}$ yields a sum of terms involving a free-index λ vector, and for each term, the derivative order on the ξ vectors are always less then r. With these considerations, the expression in (40) turns into a sum in which each term either involves a free-index λ vector or a λ^μ contraction. But, for these terms, the order of covariant derivatives acting on the ξ vector is always less than r. For the latter kind of terms, one can repeat this ongoing procedure until to the point of only having terms involving a free-index λ vector, and so no λ^μ contractions. The procedure that we discussed for (40) can be applicable to the (41) contraction pattern for which the only change will be the application of (27) instead of (29). Similarly, the analysis of a generic term involving the rth order covariant derivatives acting on $\partial_\mu V$ instead of ξ_μ is exactly the same, as was noted before.

As a result, the λ^μ contraction of a generic term involving the rth-order covariant derivative of either the ξ vector or the ∂V vector turns into a sum involving terms satisfying:

- each term involves a free-index λ vector,
- in each term, the derivative order acting on ξ or ∂V vectors is always less than r.

These were the aimed properties.

With this result, let us discuss the contractions in $E_{\mu\nu}$ or more explicitly,

$$E_{\mu\nu}=[(g^{-1})^{s-1}\mathcal{E}_{\mu_1\ldots\mu_s}]_{\mu\nu}\,, \tag{52}$$

where g^{-1} represents the inverse metric. It is clear that any nonzero contraction of $2(n_0+m)$ number of λ vectors in (38) with the other tensorial parts involving derivatives of ξ and ∂V vectors always produces a free-index λ vector and reduces the derivative order. Thus, after every nonzero λ contraction, the number of free-index λ vectors is *preserved* as $2(n_0+m)$. Obviously, one cannot avoid having a nonzero contraction once one reduces the $(0,s)$-rank tensor $\mathcal{E}_{\mu_1\ldots\mu_s}$ to a $(0,2n_0+2m)$-rank tensor, whose free indices are only on the λ vectors, and $E_{\mu\nu}$ takes the form

$$E_{\mu\nu}=[(g^{-1})^{n_0+m-1}\lambda^{2(n_0+m)}]_{\mu\nu}\,, \tag{53}$$

which is zero for $n_0+m>1$. After this observation, the only remaining possibility of having a nonzero two-tensor out of $\mathcal{E}_{\mu_1\ldots\mu_s}$ is to have only two λ one-forms from the *outset*, so either $n_0=1$ or $m=1$, implying the presence of only one Riemann tensor in $\mathcal{E}_{\mu_1\ldots\mu_s}$. Thus, the generic forms of a *nonzero* two-tensor are

$$[R]_{\mu\nu}\,,\qquad [\nabla^n R]_{\mu\nu}\,, \tag{54}$$

where n is even and $[R]_{\mu\nu}$ just represents the Ricci tensor while the second term represents a two-tensor contraction of

$$\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_n}R_{\nu_1\nu_2\nu_3\nu_4}. \tag{55}$$

In analyzing two-tensor contractions of (55), the important observation is that in the process of obtaining a nonzero two-tensor, one can freely change the order of the covariant derivatives by using the Ricci identity since all the additional terms involving a second Riemann tensor just yield a zero at the two-tensor level as we just proved[10]. In obtaining a nonzero two-tensor out of (55), one can have two contraction possibilities either

$$\nabla_{\mu_1}\ldots\nabla^{\nu_1}\ldots\nabla^{\nu_3}\ldots\nabla_{\mu_n}R_{\nu_1\nu_2\nu_3\nu_4}, \tag{56}$$

or

$$\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_n}R_{\nu_1\nu_2}. \tag{57}$$

For both of them, the following contractions of the covariant derivatives are among themselves. Because $\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_n}R$ is zero as the Ricci scalar R is constant and

$$\nabla_{\mu_1}\ldots\nabla^{\nu_1}\ldots\nabla_{\mu_n}R_{\nu_1\nu_2}, \tag{58}$$

yields a zero since one can change the orders of covariant derivatives until one obtains $\nabla_{\mu_1}\ldots\nabla_{\mu_n}\nabla_{\nu_2}R$. In (56), one can change the order of derivatives by the Ricci identity to obtain

$$\nabla_{\mu_1}\ldots\nabla_{\mu_n}\nabla^{\nu_1}\nabla^{\nu_3}R_{\nu_1\nu_2\nu_3\nu_4}=\nabla_{\mu_1}\ldots\nabla_{\mu_n}\left(\Box-\frac{R}{D-1}\right)S_{\nu_2\nu_4}, \tag{59}$$

where we used (25). Note also that (57) becomes

$$\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_n}R_{\nu_1\nu_2}=\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_n}S_{\nu_1\nu_2}. \tag{60}$$

The remaining free-indices in the covariant derivatives of (59) and (60) can be rearranged such that one has

$$\Box^{\frac{n-2}{2}}\left(\Box-\frac{R}{D-1}\right)S_{\nu_2\nu_4}, \tag{61}$$

and $\Box^{n/2}S_{\nu_1\nu_2}$, respectively. Note that for a change of order involving the first two derivatives, it may seem that there is a possibility of having additional nonzero terms due to the metric part in (19). But, one never needs such a change since for a term in the form

$$\nabla^{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_1}\ldots\nabla_{\mu_n}S_{\nu_1\nu_2}, \tag{62}$$

one may only move ∇_{μ_1} to obtain

$$\Box\nabla_{\mu_2}\ldots\nabla_{\mu_n}S_{\nu_1\nu_2}. \tag{63}$$

As a result, the generic two-tensor $E_{\mu\nu}$ constructed from any number of Riemann tensors and its covariant derivatives can be written as a sum of the metric, $S_{\mu\nu}$, and higher derivatives of $S_{\mu\nu}$ in the form $\Box^n S_{\mu\nu}$. This proves the theorem. □

In the appendix, we give another, mathematical induction based, proof of the theorem.

[10] Note that for an order change involving the first two derivatives, there is a possibility of having an additional nonzero term in the form $[\nabla^{n-2}R]_{\mu\nu}$ due to the metric part in the Riemann tensor (19).

4. Conclusions

We have shown that the Kerr-Schild–Kundt class of metrics, defined by the relations (2) and (3), are universal in the sense that they solve the most general quantum-corrected source-free gravity equations based on the metric tensor, the Riemann tensor and its arbitrary number of covariant derivatives and their powers. Our proof here boils down to showing that the generic two-tensor built out of the contractions of the Riemann tensor and its covariant derivatives can be written as a symmetric, covariantly-conserved, two-tensor $E_{\mu\nu}$ for the KSK-class in the form

$$E_{\mu\nu} = e g_{\mu\nu} + \sum_{n=0}^{N} a_n \Box^n S_{\mu\nu}, \tag{64}$$

where e and a_n are parameters, constants, of the theory. One further reduction gives the product of scalar wave type equations (10), generically one of them is massless and the rest are massive. The massless one corresponds to the Einstein's theory, and the massive ones correspond to quadratic gravity. Of course, one must still solve these equations to actually find explicit solutions: namely, one must determine the metric function V. We have not done this in the current work because, earlier, we already gave examples of these metrics such as the AdS-plane and AdS-spherical waves as solutions to quadratic and generic gravity theories [13, 15, 17]. In [20, 21], we give a systematic way of constructing solutions, such universal metrics, from curves living in one less dimension and extend the discussion to the de Sitter case.

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Appendix A. Alternative proof by induction

In this appendix, for a second proof of theorem 1, alternative to the one given in the bulk of the paper, we give necessary recursion relations satisfied by the tensors in KSK spacetimes. A generic two-tensor constructed out of the Riemann tensor and its covariant derivatives can be represented as

$$E_{\mu\nu} \equiv [R^{n_0}(\nabla^{n_1}R)(\nabla^{n_2}R)\dots(\nabla^{n_m}R)]_{\mu\nu}\,, \tag{A.1}$$

where the Riemann tensor R for KSK metrics is

$$R_{\mu\alpha\nu\beta} = 4\lambda_{[\mu}\Theta_{\alpha][\beta}\lambda_{\nu]} + \frac{4\Lambda}{(D-1)(D-2)} g_{\mu[\nu}g_{\beta]\alpha}. \tag{A.2}$$

Here, $\Theta_{\alpha\beta}$ is defined as

$$\Theta_{\alpha\beta} = -\left(\nabla_\alpha\partial_\beta + \xi_{(\alpha}\partial_{\beta)} + \frac{1}{2}\xi_\alpha\xi_\beta - \frac{2}{\ell^2}g_{\alpha\beta}\right)V. \tag{A.3}$$

Assuming n_m is to be the largest integer, the $\left(0, s \equiv 4n_0 + 4m + \sum_{n_i=1}^{i=m} n_i\right)$ rank tensor,

$$\mathcal{E}_{\mu_1\dots\mu_s} \equiv [R^{n_0}(\nabla^{n_1}R)(\nabla^{n_2}R)\dots(\nabla^{n_m}R)]\,, \tag{A.4}$$

represents a sum of rank $(0,s)$ tensors which can be decomposed into $2(n_0+m)$ number of λ vectors and rank $(0, s-2n_0-2m)$ tensor structures which are built of the contractions of the following building blocks

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r}\nabla_{\mu_i}\right)\xi_{\mu_{r+1}}, \qquad \left(\prod_{i=1}^{r+2}\nabla_{\mu_i}\right)V, \qquad r=1,2,\dots,n_m. \tag{A.5}$$

We need to understand the contractions of λ with these building blocks. For this purpose, we need the following definitions:

Definition 1 (λ-reducible tensor). A tensor E of rank $(0,m)$ is called λ-reducible if it can be written as

$$E_{\mu_1\mu_2\cdots\mu_m} = \sum_{s=1}^{m}\lambda_{\mu_s}F^{(s)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{m-1}}},$$

where $(r_1, r_2, \dots, r_{m-1})$ is an *increasing* sequence constructed with the elements of $\{1,2,\dots,m\}\setminus\{s\}$ (where the notation denotes the set-theoretic difference, that is s is omitted from the set), and $F^{(s)}$ tensors are rank $(0,m-1)$ tensors containing no free λ vectors.

Definition 2 (λ-weight of a tensor). A tensor E of rank $(0,m)$ has the λ-weight n if it can be written as a linear combination of $(0,m)$ rank tensors which can be decomposed into n number of λ vectors and rank $(0,m-n)$ tensors $F^{(s)}$ which are *not* λ-reducible, that is

$$E_{\mu_1\mu_2\cdots\mu_m} = \sum_{s=1}^{N}\lambda_{\mu_{k_1^s}}\lambda_{\mu_{k_2^s}}\dots\lambda_{\mu_{k_n^s}}F^{(s)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{m-n}}}, \tag{A.6}$$

where N is the number of the n-element subsets of $\{1,2,\dots,m\}$, that is $N=\binom{m}{n}$, s is the label for each of these n-element subsets such that for each s, $\{k_1^s, k_2^s, \dots, k_n^s\}$ is one of these n-element subsets, and $(r_1, r_2, \dots, r_{m-n})$ is an *increasing* sequence constructed with the elements of $\{1,2,\dots,m\}\setminus\{k_1^s, k_2^s, \dots, k_n^s\}$.

Remark. All the F tensors in the following discussions are assumed to be *not* λ-reducible.

Example 1. The Weyl tensor $C_{\mu\alpha\nu\beta}$ has the λ-weight 2 since it reads for the KSK class as

$$C_{\mu\alpha\nu\beta} = \lambda_\mu\lambda_\nu\,\Omega_{\alpha\beta} + \lambda_\alpha\lambda_\beta\,\Omega_{\mu\nu} - \lambda_\mu\lambda_\beta\,\Omega_{\alpha\nu} - \lambda_\alpha\lambda_\nu\,\Omega_{\mu\beta}\,, \tag{A.7}$$

or

$$C_{\mu_1\mu_2\mu_3\mu_4} = \sum_{s=1}^{6}\lambda_{\mu_{k_1^s}}\lambda_{\mu_{k_2^s}}F^{(s)}_{\mu_{r_1}\mu_{r_2}}, \tag{A.8}$$

where for the subsets $\{1,2\}$ and $\{3,4\}$, $F^{(s)}_{\mu_{r_1}\mu_{r_2}}=0$, while for the others, $F^{(s)}_{\mu_{r_1}\mu_{r_2}} = \Omega_{\mu_{r_1}\mu_{r_2}}$. In addition, (r_1, r_2) is an increasing sequence constructed with the elements of $\{1,2,3,4\}\setminus\{k_1^s, k_2^s\}$.

As another example, the traceless-Ricci tensor $S_{\mu\nu} = \rho\lambda_\mu\lambda_\nu$ has the λ-weight 2.

Definition 3 (λ-conserving tensor). Let E be a λ-weight n tensor of rank $(0,m)$. The E tensor is λ-conserving if its λ-weight increases by one after each nonzero contraction with λ.

Example 2. $\nabla_{\mu_1}\xi_{\mu_2}$ is a λ-weight conserving tensor since under a contraction with one λ vector, its λ-weight becomes 1 as

$$\lambda^{\mu_1}\nabla_{\mu_1}\xi_{\mu_2} = -\lambda_{\mu_2}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{R}{D(D-1)}\right), \tag{A.9}$$

and a further λ-contraction yields zero. Also, $\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}$ is λ-weight conserving since under a contraction with one λ vector, its λ-weight becomes 1 as can be seen from all the nonvanishing contractions

$$\begin{aligned}\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = &-\frac{1}{2}\lambda_{\mu_2}\left[\xi_{\mu_3}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{R}{D(D-1)}\right) + \xi^{\mu_1}\nabla_{\mu_1}\xi_{\mu_3}\right]\\ &+\lambda_{\mu_3}\left(-\frac{1}{2}\xi^{\mu_1}\nabla_{\mu_2}\xi_{\mu_1} + \frac{R}{D(D-1)}\xi_{\mu_2}\right),\end{aligned} \tag{A.10}$$

$$\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = -\frac{1}{2}\lambda_{\mu_1}\left[\xi_{\mu_3}\left(\frac{1}{4}\xi^{\mu_2}\xi_{\mu_2} - \frac{R}{D(D-1)}\right) + \xi^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3}\right] - \frac{1}{2}\lambda_{\mu_3}\xi^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2}, \tag{A.11}$$

$$\lambda^{\mu_3}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = -\frac{1}{2}\lambda_{\mu_1}\left(\xi^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3} + \frac{1}{2}\xi_{\mu_2}\xi^{\mu_3}\xi_{\mu_3}\right) - \lambda_{\mu_2}\xi^{\mu_3}\nabla_{\mu_1}\xi_{\mu_3}, \tag{A.12}$$

and again a further λ-contraction with any of the above contraction patterns of $\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}$ yields zero.

Lemma 1. *For a λ-conserving tensor of rank $(0,m)$ and λ-weight n, the maximum number of nonzero λ contractions is $(m-n)/2$ for even $(m-n)$ and $(m-n-1)/2$ for odd $(m-n)$.*

Proof. Under p number of λ contractions, a λ-conserving E tensor of λ-weight n and rank $(0,m)$ yields a λ-weight $(n+p)$ tensor of rank $(0,m-p)$ in the form

$$\lambda^{\mu_{j_1}}\lambda^{\mu_{j_2}}\ldots\lambda^{\mu_{j_p}}E_{\mu_1\mu_2\cdots\mu_m} = \sum_{s=1}^{N}\lambda_{\mu_{k_1^s}}\lambda_{\mu_{k_2^s}}\ldots\lambda_{\mu_{k_{n+p}^s}}F^{(s)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{m-n-2p}}}, \tag{A.13}$$

where $\{j_1,j_2,\ldots,j_p\}$ is a subset of $\{1,2,\ldots,m\}$, N is the number of the $(n+p)$-element subsets of $\{1,2,\ldots,m\}\setminus\{j_1,j_2,\ldots,j_p\}$, that is $N=\binom{m-p}{n+p}$, s is the label for these $(n+p)$-element subsets such that for each s, $\{k_1^s,k_2^s,\ldots,k_{n+p}^s\}$ is one of these subsets, and $(r_1,r_2,\ldots,r_{m-n-2p})$ is an increasing sequence constructed from $\{1,2,\ldots,m\}\setminus\left(\{j_1,j_2,\ldots,j_p\}\cup\{k_1^s,k_2^s,\ldots,k_{n+p}^s\}\right)$. Here, we assumed $m-n>2p$ as must be the case in (A.13).

This result implies that the maximum number of λ contractions is $(m-n)/2$ for even $(m-n)$ and $(m-n-1)/2$ for odd $(m-n)$. Then, one gets the following results, respectively, for even and odd $(m-n)$;

$$\left(\prod_{s=1}^{\frac{m-n}{2}}\lambda^{\mu_{j_s}}\right)E_{\mu_1\mu_2\cdots\mu_m}=\left(\prod_{s=1}^{\frac{m+n}{2}}\lambda_{\mu_s}\right)F,$$

and

$$\left(\prod_{s=1}^{\frac{m-n-1}{2}}\lambda^{\mu_{j_s}}\right)E_{\mu_1\mu_2\cdots\mu_m}=\sum_{s=1}^{\frac{m+n-1}{2}}F_{\mu_s}\lambda_{\mu_{r_1}}\lambda_{\mu_{r_2}}\ldots\lambda_{\mu_{r_{(m+n-1)/2}}},$$

where $\{r_1,r_2,\ldots,r_{(m+n-1)/2}\}=\{1,2,\ldots,m\}\setminus\{j_1,j_2,\ldots,j_{(m-n-1)/2},s\}$. Here, note that for a λ-conserving tensor E, $\lambda^{\mu_s}F_{\mu_s}$ should be zero. This proves the lemma. □

Lemma 2. *For a λ-conserving tensor of rank $(0,m)$ and λ-weight n, the contractions among its indices do not change the λ-weight of the tensor.*

Proof. A λ-weight n tensor E of rank $(0,m)$ has the form

$$E_{\mu_1\mu_2\cdots\mu_m}=\sum_{s=1}^{N}\lambda_{\mu_{k_1^s}}\lambda_{\mu_{k_2^s}}\ldots\lambda_{\mu_{k_n^s}}F^{(s)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{m-n}}},$$

where $(r_1,r_2,\ldots,r_{m-n})$ is an increasing sequence constructed with the elements of $\{1,2,\ldots,m\}\setminus\{k_1^s,k_2^s,\ldots,k_n^s\}$. The λ-weight zero tensors $F^{(s)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{m-n}}}$ are λ-conserving since E is λ-conserving. Then, contractions among the indices of E can be either λ-λ contraction, or λ-F contraction, or a contraction among the indices of the F tensor. The λ-λ contraction is zero. A contraction among the indices of the F tensor surely does not change the λ-weight. Finally, the result of each λ-F contraction increases the λ-weight by one, so the total λ-weight still remains as n. □

Theorem 2. *The rank $(0,n)$ tensor $\nabla^{n-1}\xi$ is λ-conserving.*

To prove this theorem, we need the following two lemmas below. Let us introduce the indices of $\nabla^{n-1}\xi$ as

$$\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\equiv\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{n-1}}\xi_{\mu_n}. \tag{A.14}$$

To show that $\nabla^{n-1}\xi$ is λ-conserving, first let us prove that $\lambda^{\mu_j}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}$, where j takes a value from $\{1,2,\ldots,n\}$, is λ-reducible by using mathematical induction.

Lemma 3. *$\lambda^{\mu_j}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}$, where j takes a value from $\{1,2,\ldots,n\}$, is λ-reducible.*

Proof. As the basis set of identities, we know that $\nabla_\mu\lambda_\nu=\xi_{(\mu}\lambda_{\nu)}$ and ξ satisfies the identities

$$\lambda^{\mu_1}\nabla_{\mu_1}\xi_{\mu_2}=-\lambda_{\mu_2}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1}-\frac{R}{D(D-1)}\right), \tag{A.15}$$

and

$$\lambda^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2}=-\frac{1}{2}\lambda_{\mu_1}\xi^{\mu_2}\xi_{\mu_2}. \tag{A.16}$$

For mathematical induction, as the first step, the $n=2$ case given above is sufficient. But, we will include the $n=3$ case, to obtain some insight which will be useful in further calculations. Then, moving to the mathematical induction proof:

1. For $n=3$, $\lambda^{\mu_j}\left(\prod_{i=1}^{2}\nabla_{\mu_i}\right)\xi_{\mu_3}$ involves the contraction patterns

$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}, \qquad \lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}, \qquad \lambda^{\mu_3}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}. \tag{A.17}$$

The first contraction pattern reduces to the second one by interchanging the order of the derivatives as

$$\begin{aligned}\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} &= \lambda^{\mu_1}\left[\nabla_{\mu_1},\nabla_{\mu_2}\right]\xi_{\mu_3}+\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\xi_{\mu_3}\\ &= \lambda^{\mu_1}R_{\mu_1\mu_2\mu_3\mu_4}\xi^{\mu_4}+\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\xi_{\mu_3},\end{aligned} \tag{A.18}$$

and from (B27) of [17], one has

$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = \frac{R}{D(D-1)}\lambda_{\mu_3}\xi_{\mu_2}+\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\xi_{\mu_3}. \tag{A.19}$$

Thus, if the second contraction pattern is λ-reducible, then so is the first one. Moving to the second contraction pattern which becomes

$$\begin{aligned}\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} &= \nabla_{\mu_1}(\lambda^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3})-(\nabla_{\mu_1}\lambda^{\mu_2})\nabla_{\mu_2}\xi_{\mu_3}\\ &= \nabla_{\mu_1}(\lambda^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3})-\frac{1}{2}\xi_{\mu_1}\lambda^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3}-\frac{1}{2}\lambda_{\mu_1}\xi^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3},\end{aligned} \tag{A.20}$$

and using the identity for $\lambda^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3}$, one obtains

$$\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = -\frac{1}{2}\lambda_{\mu_1}\left[\xi_{\mu_3}\left(\frac{1}{4}\xi^{\mu_2}\xi_{\mu_2}-\frac{R}{D(D-1)}\right)+\xi^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3}\right]-\frac{1}{2}\lambda_{\mu_3}\xi^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2}, \tag{A.21}$$

which is λ-reducible. With this result, the first contraction is also λ-reducible and takes the form

$$\begin{aligned}\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = &-\frac{1}{2}\lambda_{\mu_2}\left[\xi_{\mu_3}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1}-\frac{R}{D(D-1)}\right)+\xi^{\mu_1}\nabla_{\mu_1}\xi_{\mu_3}\right]\\ &+\lambda_{\mu_3}\left(-\frac{1}{2}\xi^{\mu_1}\nabla_{\mu_2}\xi_{\mu_1}+\frac{R}{D(D-1)}\xi_{\mu_2}\right).\end{aligned} \tag{A.22}$$

Lastly, the third contraction pattern can be written as

$$\begin{aligned}\lambda^{\mu_3}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} &= \nabla_{\mu_1}(\lambda^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3})-(\nabla_{\mu_1}\lambda^{\mu_3})\nabla_{\mu_2}\xi_{\mu_3}\\ &= \nabla_{\mu_1}(\lambda^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3})-\frac{1}{2}\xi_{\mu_1}\lambda^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3}-\frac{1}{2}\lambda_{\mu_1}\xi^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3},\end{aligned} \tag{A.23}$$

and using the identity for $\lambda^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2}$, one obtains

$$\lambda^{\mu_3}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = -\frac{1}{2}\lambda_{\mu_1}\left(\xi^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3}+\frac{1}{2}\xi_{\mu_2}\xi^{\mu_3}\xi_{\mu_3}\right)-\lambda_{\mu_2}\xi^{\mu_3}\nabla_{\mu_1}\xi_{\mu_3}, \tag{A.24}$$

which is also λ-reducible. In summary, $\lambda^{\mu_j}\left(\prod_{i=1}^{2}\nabla_{\mu_i}\right)\xi_{\mu_3}$ is λ-reducible as

$$\lambda^{\mu_j}\left(\prod_{i=1}^{2}\nabla_{\mu_i}\right)\xi_{\mu_3}=\sum_{\substack{k=1\\(k\neq j)}}^{3}\lambda_{\mu_k}E^{(k,3,j)}_{\mu_{m_1}},\qquad m_1\in\{1,2,3\}\setminus\{j,k\},\tag{A.25}$$

where $E^{(k,3,j)}_{\mu_{m_1}}$ are built from one-form contractions of the building blocks;

$$\xi_{\mu_1},\qquad \nabla_{\mu_1}\xi_{\mu_2}.\tag{A.26}$$

2. Assume that $\lambda^{\mu_j}\left(\prod_{i=1}^{n-2}\nabla_{\mu_i}\right)\xi_{\mu_{n-1}}$ is λ-reducible for all j in $\{1,2,\ldots,n-1\}$ as

$$\lambda^{\mu_j}\left(\prod_{i=1}^{n-2}\nabla_{\mu_i}\right)\xi_{\mu_{n-1}}=\sum_{\substack{k=1\\(k\neq j)}}^{n-1}\lambda_{\mu_k}E^{(k,n-1,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}},\tag{A.27}$$

where $(m_1,m_2,\ldots,m_{n-3})$ is an *increasing* sequence constructed with the elements of $\{1,2,\ldots,n-1\}\setminus\{j,k\}$. The $E^{(k,n-1,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}$ tensors are built from the rank $(0,n-3)$ contractions of the building blocks;

$$g_{\mu_1\mu_2},\qquad \xi_{\mu_1},\qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r},\qquad r=2,3,\ldots,n-2.\tag{A.28}$$

3. Then, we must show that (A.27) holds also for $n\to n+1$, that is $\lambda^{\mu_j}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}$ is λ-reducible for all j in $\{1,2,\ldots,n\}$ as

$$\lambda^{\mu_j}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}=\sum_{\substack{k=1\\(k\neq j)}}^{n}\lambda_{\mu_k}E^{(k,n,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-2}}},\tag{A.29}$$

where $(m_1,m_2,\ldots,m_{n-2})$ is an *increasing* sequence constructed with the elements of $\{1,2,\ldots,n\}\setminus\{j,k\}$. The $E^{(k,n,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-2}}}$ tensors[11] are built from the rank $(0,n-2)$ contractions of the building blocks;

$$g_{\mu_1\mu_2},\qquad \xi_{\mu_1},\qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r},\qquad r=2,3,\ldots,n-1.\tag{A.30}$$

To show this, first note that the contraction pattern for $j=1$, that is

$$\lambda^{\mu_1}\nabla_{\mu_1}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n},\tag{A.31}$$

can be reduced to the $j=2$ term added with some terms involving the $(n-2)$ order term $\nabla^{n-3}\xi$ after changing the order of the first two covariant derivatives as

[11] It may seem that we label $E^{(k,n,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-2}}}$ with the dummy index μ_j, but in fact the j label represents the position of the covariant derivative whose index is contracted with the index of the λ vector. In this way, the k label represents the position of the index of λ_{μ_k} between the indices on the left-hand side.

$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \lambda^{\mu_1}\left[\nabla_{\mu_1},\nabla_{\mu_2}\right]\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} + \lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}$$
$$= \lambda^{\mu_1}\sum_{s=3}^{n-1} R_{\mu_1\mu_2\mu_s}{}^{\mu_{n+1}}\left(\prod_{i_1=3}^{s-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_{n+1}}\left(\prod_{i_2=s+1}^{n-1}\nabla_{\mu_{i_2}}\right)\xi_{\mu_n}$$
$$+ \lambda^{\mu_1}R_{\mu_1\mu_2\mu_n}{}^{\mu_{n+1}}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_{n+1}}$$
$$+ \lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}, \tag{A.32}$$

where using $\lambda^{\mu_1}R_{\mu_1\mu_2\mu_3\mu_4} = \frac{R}{D(D-1)}(\lambda_{\mu_3}g_{\mu_2\mu_4} - \lambda_{\mu_4}g_{\mu_2\mu_3})$, one arrives at

$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}$$
$$= \frac{R}{D(D-1)}\left[\lambda_{\mu_n}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_2} + \sum_{s=3}^{n-1}\lambda_{\mu_s}\left(\prod_{i_1=3}^{s-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_2}\left(\prod_{i_2=s+1}^{n-1}\nabla_{\mu_{i_2}}\right)\xi_{\mu_n}\right]$$
$$-\frac{R}{D(D-1)}g_{\mu_2\mu_n}\lambda^{\mu_{n+1}}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_{n+1}}$$
$$-\frac{R}{D(D-1)}\sum_{s=3}^{n-1}g_{\mu_2\mu_s}\lambda^{\mu_{n+1}}\left(\prod_{i_1=3}^{s-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_{n+1}}\left(\prod_{i_2=s+1}^{n-1}\nabla_{\mu_{i_2}}\right)\xi_{\mu_n}$$
$$+\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}. \tag{A.33}$$

The first line is λ-reducible and involves the $n-2$ order term $\nabla^{n-3}\xi$, the second and the third lines involve all one-λ contraction patterns of $\nabla^{n-3}\xi$, that is

$$\lambda^{\nu_j}\left(\prod_{i=1}^{n-3}\nabla_{\nu_i}\right)\xi_{\nu_{n-2}}, \tag{A.34}$$

with all possible j's from 1 to $n-2$, and the last line is the $j=2$ contraction pattern of $\nabla^{n-1}\xi$. Assuming $\lambda^{\nu_j}\left(\prod_{i=1}^{n-3}\nabla_{\nu_i}\right)\xi_{\nu_{n-2}}$ is also λ-reducible, then the $j=1$ contraction pattern of $\nabla^{n-1}\xi$ is λ-reducible if and only if the $j=2$ contraction pattern of $\nabla^{n-1}\xi$ is λ-reducible. Let us move on to the analysis of the $1<j\leqslant n$ contraction patterns of $\nabla^{n-1}\xi$ and let us write

$$\lambda^{\mu_j}\nabla_{\mu_1}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \nabla_{\mu_1}\left[\lambda^{\mu_j}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right] - \left(\nabla_{\mu_1}\lambda^{\mu_j}\right)\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}$$
$$= \nabla_{\mu_1}\left[\lambda^{\mu_j}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right]$$
$$-\frac{1}{2}\xi_{\mu_1}\lambda^{\mu_j}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} - \frac{1}{2}\lambda_{\mu_1}\xi^{\mu_j}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}, \tag{A.35}$$

where the last term is already λ-reducible while the first and the second terms involve the order $n-1$ term $\nabla^{n-2}\xi$ which, from the assumption (A.27), has the form[12]

$$\lambda^{\mu_j}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \sum_{\substack{k=2\\(k\neq j)}}^{n}\lambda_{\mu_k}E^{(k-1,n-1,j-1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}, \tag{A.36}$$

where $(m_1, m_2, \ldots, m_{n-3})$ is an increasing sequence constructed from $\{2,3,\ldots,n\}\setminus\{j,k\}$. Using this form, one finds

$$\lambda^{\mu_j}\nabla_{\mu_1}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \frac{1}{2}\lambda_{\mu_1}\left[\sum_{\substack{k=2\\(k\neq j)}}^{n}\xi_{\mu_k}E^{(k-1,n-1,j-1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}} - \xi^{\mu_j}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right] + \sum_{\substack{k=2\\(k\neq j)}}^{n}\lambda_{\mu_k}\nabla_{\mu_1}E^{(k-1,n-1,j-1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}, \tag{A.37}$$

so the $1<j\leqslant n$ contraction patterns of $\nabla^{n-1}\xi$ are λ-reducible. In addition, the λ-reducibility of the $j=2$ pattern implies the λ-reducibility of the $j=1$ pattern. □

Lemma 4. *The $E^{(k,n,j)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}}$ tensors can be recursively obtained from the E tensors of the lower orders.*

Proof. For the $1<j\leqslant n$ contraction patterns of $\nabla^{n-1}\xi$, we just need to compare (A.37) with

$$\lambda^{\mu_j}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \sum_{\substack{k=1\\(k\neq j)}}^{n}\lambda_{\mu_k}E^{(k,n,j)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}}, \tag{A.38}$$

where $(r_1, r_2, \ldots, r_{n-2})$ is an increasing sequence constructed from $\{1,2,\ldots,n\}\setminus\{j,k\}$, one finds

$$E^{(1,n,j)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}} = \frac{1}{2}\sum_{\substack{k=2\\(k\neq j)}}^{n}\xi_{\mu_k}E^{(k-1,n-1,j-1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}} - \frac{1}{2}\xi^{\mu_j}\left(\prod_{i=2}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}, \tag{A.39}$$

$$k\geqslant 2 \quad\Rightarrow\quad E^{(k,n,j)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}} = \nabla_{\mu_1}E^{(k-1,n-1,j-1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}, \tag{A.40}$$

where $(m_1, m_2, \ldots, m_{n-3})$ is an increasing sequence constructed from $\{2,3,\ldots,n\}\setminus\{j,k\}$.

For the $j=1$ contraction pattern of $\nabla^{n-1}\xi$, we need to make the λ-reducibility assumption for the $(n-2)$th order term $\nabla^{n-3}\xi$ more explicit and assume the form

[12] Note that in this form, we only updated the superscript of $E^{(k,n-1,j)}$ to $E^{(k-1,n-1,j-1)}$ during the change in the range of i (and so in k), because the first and the third labels of $E^{(k,n-1,j)}$ correspond to the position of the contracted index and the position of the index of λ_{μ_k} between the indices on the left-hand side. With this update, the labeling still corresponds to the correct terms in the lower order term. This enabled us to relate $E^{(k,n,j)}$'s to $E^{(k,n-1,j)}$'s.

$$\lambda^{\mu_j}\left(\prod_{i=1}^{n-3}\nabla_{\mu_i}\right)\xi_{\mu_{n-2}}=\sum_{\substack{k=1\\(k\neq j)}}^{n-2}\lambda_{\mu_k}E^{(k,n-2,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-4}}}, \tag{A.41}$$

where $(m_1, m_2, \ldots, m_{n-4})$ is an increasing sequence constructed from $\{1, 2, \ldots, n-2\} \setminus \{j, k\}$. The rank $(0, n-4)$ tensors $E^{(k,n-2,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-4}}}$ are built from the rank $(0, n-4)$ contractions of the building blocks;

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r}, \qquad r=2,3,\ldots,n-3. \tag{A.42}$$

Then, the terms

$$\lambda^{\mu_{n+1}}\left(\prod_{i_1=3}^{s-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_{n+1}}\left(\prod_{i_2=s+1}^{n-1}\nabla_{\mu_{i_2}}\right)\xi_{\mu_n}\Rightarrow\lambda^{\mu_s}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}, \tag{A.43}$$

(where we used '$\Rightarrow$' with the meaning 'can be considered as' because it is not possible to put the right-hand-side term back into (A.33); however, considering it makes sense as we just want to use (A.41),) and

$$\lambda^{\mu_{n+1}}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_{n+1}}=\lambda^{\mu_s}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_s}, \tag{A.44}$$

appearing in the $j=1$ contraction pattern of $\nabla^{n-1}\xi$, that is (A.33), can be written as

$$\lambda^{\mu_s}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}=\sum_{\substack{k=3\\(k\neq s)}}^{n}\lambda_{\mu_k}E^{(k-2,n-2,s-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}}, \tag{A.45}$$

where $(t_1, t_2, \ldots, t_{n-4})$ is an increasing sequence constructed from $\{3, 4, \ldots, n\} \setminus \{s, k\}$. In addition, using the result for the $j=2$ contraction pattern of $\nabla^{n-1}\xi$, that is

$$\begin{aligned}\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}=&\frac{1}{2}\lambda_{\mu_1}\left[\sum_{k=3}^{n}\xi_{\mu_k}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}-\xi^{\mu_2}\nabla_{\mu_2}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right]\\&+\sum_{k=3}^{n}\lambda_{\mu_k}\nabla_{\mu_1}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}},\end{aligned} \tag{A.46}$$

where $(m_1, m_2, \ldots, m_{n-3})$ is an increasing sequence constructed from $\{3, 4, \ldots, n\} \setminus \{k\}$, the last term in (A.33) can be written as

$$\begin{aligned}\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}=&\frac{1}{2}\lambda_{\mu_2}\left[\sum_{k=3}^{n}\xi_{\mu_k}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}-\xi^{\mu_1}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right]\\&+\sum_{k=3}^{n}\lambda_{\mu_k}\nabla_{\mu_2}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}},\end{aligned} \tag{A.47}$$

where $(m_1, m_2, \ldots, m_{n-3})$ is an increasing sequence constructed from $\{3, 4, \ldots, n\} \setminus \{k\}$. Using (A.45) and (A.47) in (A.33), one obtains

$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \frac{R}{D(D-1)}\left[\lambda_{\mu_n}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_2} + \sum_{s=3}^{n-1}\lambda_{\mu_s}\left(\prod_{i_1=3}^{s-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_2}\left(\prod_{i_2=s+1}^{n-1}\nabla_{\mu_{i_2}}\right)\xi_{\mu_n}\right]$$
$$- \frac{R}{D(D-1)}\left[g_{\mu_2\mu_n}\sum_{k=3}^{n-1}\lambda_{\mu_k}E^{(k-2,n-2,n-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}} + \sum_{s=3}^{n-1}g_{\mu_2\mu_s}\sum_{\substack{k=3\\(k\neq s)}}^{n}\lambda_{\mu_k}E^{(k-2,n-2,s-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}}\right]$$
$$+ \frac{1}{2}\lambda_{\mu_2}\left[\sum_{k=3}^{n}\xi_{\mu_k}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}} - \xi^{\mu_1}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right] + \sum_{k=3}^{n}\lambda_{\mu_k}\nabla_{\mu_2}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}. \tag{A.48}$$

Here, we can change the order of summations in the second term of the second line as

$$\sum_{s=3}^{n-1}g_{\mu_2\mu_s}\sum_{\substack{k=3\\(k\neq s)}}^{n}\lambda_{\mu_k}E^{(k-2,n-2,s-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}} = \sum_{k=3}^{n}\lambda_{\mu_k}\sum_{\substack{s=3\\(s\neq k)}}^{n-1}g_{\mu_2\mu_s}E^{(k-2,n-2,s-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}}, \tag{A.49}$$

then with this result, one has

$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \frac{1}{2}\lambda_{\mu_2}\left[\sum_{k=3}^{n}\xi_{\mu_k}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}} - \xi^{\mu_1}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right]$$
$$+ \sum_{k=3}^{n-1}\lambda_{\mu_k}\left[\frac{R}{D(D-1)}\left(\prod_{i_1=3}^{k-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_2}\left(\prod_{i_2=k+1}^{n-1}\nabla_{\mu_{i_2}}\right)\xi_{\mu_n} + \nabla_{\mu_2}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}\right]$$
$$- \frac{R}{D(D-1)}\sum_{k=3}^{n-1}\lambda_{\mu_k}\left(g_{\mu_2\mu_n}E^{(k-2,n-2,n-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}} + \sum_{\substack{s=3\\(s\neq k)}}^{n-1}g_{\mu_2\mu_s}E^{(k-2,n-2,s-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}}\right)$$
$$+ \lambda_{\mu_n}\left\{\frac{R}{D(D-1)}\left[\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_2} - \sum_{s=3}^{n-1}g_{\mu_2\mu_s}E^{(n-2,n-2,s-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}}\right] + \nabla_{\mu_2}E^{(n-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}\right\}, \tag{A.50}$$

The third line of this result can be written as

$$g_{\mu_2\mu_n}E^{(k-2,n-2,n-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}} + \sum_{\substack{s=3\\(s\neq k)}}^{n-1}g_{\mu_2\mu_s}E^{(k-2,n-2,s-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}} = \sum_{\substack{s=3\\(s\neq k)}}^{n}g_{\mu_2\mu_s}E^{(k-2,n-2,s-2)}_{\mu_{t_1}\mu_{t_2}\cdots\mu_{t_{n-4}}}, \tag{A.51}$$

and then reordering the terms and rewriting the t indices as m indices, the final form becomes

$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \frac{1}{2}\lambda_{\mu_2}\left[\sum_{k=3}^{n}\xi_{\mu_k}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}} - \xi^{\mu_1}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right]$$
$$+\frac{R}{D(D-1)}\sum_{k=3}^{n-1}\lambda_{\mu_k}\left[\left(\prod_{i_1=3}^{k-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_2}\left(\prod_{i_2=k+1}^{n-1}\nabla_{\mu_{i_2}}\right)\xi_{\mu_n} - \sum_{\substack{s=3\\(s\neq k)}}^{n} g_{\mu_2\mu_s}E^{(k-2,n-2,s-2)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-4}}}\right]$$
$$+\sum_{k=3}^{n-1}\lambda_{\mu_k}\nabla_{\mu_2}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}$$
$$+\lambda_{\mu_n}\left\{\frac{R}{D(D-1)}\left[\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_2} - \sum_{s=3}^{n-1}g_{\mu_2\mu_s}E^{(n-2,n-2,s-2)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-4}}}\right] + \nabla_{\mu_2}E^{(n-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}\right\}, \tag{A.52}$$

where $(m_1, m_2, \ldots, m_{n-3})$ and $(m_1, m_2, \ldots, m_{n-4})$ are increasing sequences constructed from$\{3, 4, \ldots, n\} \setminus \{k\}$ and $\{3, 4, \ldots, n\} \setminus \{s, k\}$, respectively[13]. Then, by comparing this result with

$$\lambda^{\mu_1}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \sum_{k=2}^{n}\lambda_{\mu_k}E^{(k,n,1)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}}, \tag{A.53}$$

where $(r_1, r_2, \ldots, r_{n-2})$ is an increasing sequence constructed from $\{2, 3, \ldots, n\} \setminus \{k\}$, one arrives at

$$E^{(2,n,1)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}} = \frac{1}{2}\left[\sum_{k=3}^{n}\xi_{\mu_k}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}} - \xi^{\mu_1}\nabla_{\mu_1}\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}\right], \tag{A.54}$$

$$2 < k < n \quad \Rightarrow \quad E^{(k,n,1)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}} = \frac{R}{D(D-1)}\left(\prod_{i_1=3}^{k-1}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_2}\left(\prod_{i_2=k+1}^{n-1}\nabla_{\mu_{i_2}}\right)\xi_{\mu_n}$$
$$-\frac{R}{D(D-1)}\sum_{\substack{s=3\\(s\neq k)}}^{n} g_{\mu_2\mu_s}E^{(k-2,n-2,s-2)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-4}}}$$
$$+\nabla_{\mu_2}E^{(k-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}, \tag{A.55}$$

$$E^{(n,n,1)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}} = \frac{R}{D(D-1)}\left[\left(\prod_{i=3}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_2} - \sum_{s=3}^{n-1}g_{\mu_2\mu_s}E^{(n-2,n-2,s-2)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-4}}}\right]$$
$$+\nabla_{\mu_2}E^{(n-1,n-1,1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}. \tag{A.56}$$

where $(m_1, m_2, \ldots, m_{n-3})$ and $(m_1, m_2, \ldots, m_{n-4})$ are increasing sequences constructed from $\{3, 4, \ldots, n\} \setminus \{k\}$ and $\{3, 4, \ldots, n\} \setminus \{s, k\}$, respectively. □

[13] Note that in the last line $k = n$.

Example 3. To apply the recursive relations (A.39) and (A.40), let us use the $n = 3$ result. For $j = 2$, one has

$$\lambda^{\mu_2}\left(\prod_{i=1}^{2}\nabla_{\mu_i}\right)\xi_{\mu_3} = \sum_{\substack{k=1\\(k\neq 2)}}^{3}\lambda_{\mu_k}E^{(k,3,2)}_{\mu_{r_1}}$$
$$\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = \lambda_{\mu_1}E^{(1,3,2)}_{\mu_3} + \lambda_{\mu_3}E^{(3,3,2)}_{\mu_1}, \tag{A.57}$$

and from (A.39) and (A.40), one has

$$E^{(1,3,2)}_{\mu_3} = \frac{1}{2}\sum_{\substack{k=2\\(k\neq 2)}}^{3}\xi_{\mu_k}E^{(k-1,2,1)} - \frac{1}{2}\xi^{\mu_2}\left(\prod_{i=2}^{2}\nabla_{\mu_i}\right)\xi_{\mu_3}$$
$$= \frac{1}{2}\xi_{\mu_3}E^{(2,2,1)} - \frac{1}{2}\xi^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3}, \tag{A.58}$$

and

$$E^{(3,3,2)}_{\mu_1} = \nabla_{\mu_1}E^{(2,2,1)}. \tag{A.59}$$

Here, $E^{(2,2,1)}$ should be obtained from $\lambda^{\mu_1}\nabla_{\mu_1}\xi_{\mu_2}$ and

$$\lambda^{\mu_1}\nabla_{\mu_1}\xi_{\mu_2} = \sum_{\substack{k=1\\(k\neq 1)}}^{2}\lambda_{\mu_k}E^{(k,2,1)} = \lambda_{\mu_2}E^{(2,2,1)}, \tag{A.60}$$

and we know that $\lambda^{\mu_1}\nabla_{\mu_1}\xi_{\mu_2}$ satisfies

$$\lambda^{\mu_1}\nabla_{\mu_1}\xi_{\mu_2} = -\lambda_{\mu_2}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{R}{D(D-1)}\right), \tag{A.61}$$

from which $E^{(2,2,1)}$ can be obtained as

$$E^{(2,2,1)} = -\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{R}{D(D-1)}\right). \tag{A.62}$$

Using this result, $E^{(1,3,2)}_{\mu_3}$ and $E^{(3,3,2)}_{\mu_1}$ become

$$E^{(1,3,2)}_{\mu_3} = -\frac{1}{2}\xi_{\mu_3}\left(\frac{1}{4}\xi^{\mu_2}\xi_{\mu_2} - \frac{R}{D(D-1)}\right) - \frac{1}{2}\xi^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3}, \tag{A.63}$$

and

$$E^{(3,3,2)}_{\mu_1} = -\frac{1}{2}\xi^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2}. \tag{A.64}$$

With these results, $\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}$ becomes

$$\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = \lambda_{\mu_1}\left[-\frac{1}{2}\xi_{\mu_3}\left(\frac{1}{4}\xi^{\mu_2}\xi_{\mu_2} - \frac{R}{D(D-1)}\right) - \frac{1}{2}\xi^{\mu_2}\nabla_{\mu_2}\xi_{\mu_3}\right] - \frac{1}{2}\lambda_{\mu_3}\xi^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2}, \tag{A.65}$$

which is the same as (A.21). Let us also apply the recursive relations (A.39) and (A.40) in the case of the $j = 3$ contraction pattern of $\nabla\nabla\xi$ for which one has

$$\lambda^{\mu_3}\left(\prod_{i=1}^{2}\nabla_{\mu_i}\right)\xi_{\mu_3} = \sum_{\substack{k=1\\(k\neq 3)}}^{3}\lambda_{\mu_k}E^{(k,3,3)}_{\mu_{r_1}} = \lambda_{\mu_1}E^{(1,3,3)}_{\mu_2} + \lambda_{\mu_2}E^{(2,3,3)}_{\mu_1}, \tag{A.66}$$

and from the recursive relations, one has

$$\begin{aligned} E^{(1,3,3)}_{\mu_2} &= \frac{1}{2}\sum_{\substack{k=2\\(k\neq 3)}}^{3}\xi_{\mu_k}E^{(k-1,2,2)} - \frac{1}{2}\xi^{\mu_3}\left(\prod_{i=2}^{2}\nabla_{\mu_i}\right)\xi_{\mu_3} \\ &= \frac{1}{2}\xi_{\mu_2}E^{(1,2,2)} - \frac{1}{2}\xi^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3}, \end{aligned} \tag{A.67}$$

and

$$E^{(2,3,3)}_{\mu_1} = \nabla_{\mu_1}E^{(1,2,2)}. \tag{A.68}$$

Here, $E^{(1,2,2)}$ should be obtained from $\lambda^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2}$ and

$$\lambda^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2} = \sum_{\substack{k=1\\(k\neq 2)}}^{2}\lambda_{\mu_k}E^{(k,2,2)} = \lambda_{\mu_1}E^{(1,2,2)}, \tag{A.69}$$

and we know that $\lambda^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2}$ satisfies

$$\lambda^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2} = -\frac{1}{2}\lambda_{\mu_1}\xi^{\mu_2}\xi_{\mu_2}, \tag{A.70}$$

from which $E^{(1,2,2)}$ can be obtained as

$$E^{(1,2,2)} = -\frac{1}{2}\xi^{\mu_2}\xi_{\mu_2}. \tag{A.71}$$

Using this result, $E^{(1,3,3)}_{\mu_2}$ and $E^{(2,3,3)}_{\mu_1}$ become

$$E^{(1,3,3)}_{\mu_2} = -\frac{1}{4}\xi_{\mu_2}\xi^{\mu_3}\xi_{\mu_3} - \frac{1}{2}\xi^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3}, \tag{A.72}$$

and

$$E^{(2,3,3)}_{\mu_1} = -\xi^{\mu_3}\nabla_{\mu_1}\xi_{\mu_3}. \tag{A.73}$$

With these results, $\lambda^{\mu_3}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}$ becomes

$$\lambda^{\mu_3}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3} = \lambda_{\mu_1}\left(-\frac{1}{4}\xi_{\mu_2}\xi^{\mu_3}\xi_{\mu_3} - \frac{1}{2}\xi^{\mu_3}\nabla_{\mu_2}\xi_{\mu_3}\right) - \lambda_{\mu_2}\xi^{\mu_3}\nabla_{\mu_1}\xi_{\mu_3}, \tag{A.74}$$

which is the same as (A.24).

Example 4. Now, let us calculate $\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_4}$ explicitly and also compute it with the recursion relations (A.54)–(A.56). This calculation demonstrates the usefulness of these recursion relations at the first nontrivial order. Thus, $\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_4}$ can be calculated in terms of $\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\nabla_{\mu_3}\xi_{\mu_4}$ as

$$\begin{aligned}\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_4} = {}& \frac{R}{D(D-1)}\Big(\lambda_{\mu_3}\nabla_{\mu_2}\xi_{\mu_4}+\lambda_{\mu_4}\nabla_{\mu_3}\xi_{\mu_2}\Big)\\ & -\frac{R}{D(D-1)}\Big(g_{\mu_2\mu_3}\lambda^{\mu_5}\nabla_{\mu_5}\xi_{\mu_4}+g_{\mu_2\mu_4}\lambda^{\mu_5}\nabla_{\mu_3}\xi_{\mu_5}\Big)\\ & +\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\nabla_{\mu_3}\xi_{\mu_4}.\end{aligned} \tag{A.75}$$

Then, calculating $\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\nabla_{\mu_3}\xi_{\mu_4}$ yields

$$\begin{aligned}\lambda^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\nabla_{\mu_3}\xi_{\mu_4} = {}& -\frac{1}{2}\lambda_{\mu_2}\left[\xi^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_3}\xi_{\mu_4}+\frac{1}{2}\xi_{\mu_3}\xi^{\mu_1}\nabla_{\mu_1}\xi_{\mu_4}+\frac{1}{2}\xi_{\mu_4}\xi^{\mu_1}\nabla_{\mu_3}\xi_{\mu_1}\right]\\ & -\frac{1}{4}\lambda_{\mu_2}\xi_{\mu_3}\xi_{\mu_4}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1}-\frac{3R}{D(D-1)}\right)\\ & -\frac{1}{2}\lambda_{\mu_3}\left[\xi^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\xi_{\mu_4}+\big(\nabla_{\mu_2}\xi^{\mu_1}\big)\nabla_{\mu_1}\xi_{\mu_4}\right]\\ & -\frac{1}{2}\lambda_{\mu_3}\left[\big(\nabla_{\mu_2}\xi_{\mu_4}\big)\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1}-\frac{R}{D(D-1)}\right)+\frac{1}{2}\xi_{\mu_4}\xi^{\mu_1}\nabla_{\mu_2}\xi_{\mu_1}\right]\\ & -\frac{1}{2}\lambda_{\mu_4}\left[\xi^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_1}+\big(\nabla_{\mu_2}\xi^{\mu_1}\big)\nabla_{\mu_3}\xi_{\mu_1}-\frac{2R}{D(D-1)}\nabla_{\mu_2}\xi_{\mu_3}\right].\end{aligned} \tag{A.76}$$

Using this result, $\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_4}$ becomes

$$\begin{aligned}\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_4} = {}& -\frac{1}{2}\lambda_{\mu_2}\left[\xi^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_3}\xi_{\mu_4}+\frac{1}{2}\xi_{\mu_3}\xi^{\mu_1}\nabla_{\mu_1}\xi_{\mu_4}+\frac{1}{2}\xi_{\mu_4}\xi^{\mu_1}\nabla_{\mu_3}\xi_{\mu_1}\right]\\ & -\frac{1}{4}\lambda_{\mu_2}\xi_{\mu_3}\xi_{\mu_4}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1}-\frac{3R}{D(D-1)}\right)\\ & -\frac{1}{2}\lambda_{\mu_3}\left[\xi^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\xi_{\mu_4}+\big(\nabla_{\mu_2}\xi^{\mu_1}\big)\nabla_{\mu_1}\xi_{\mu_4}-\frac{R}{D(D-1)}g_{\mu_2\mu_4}\xi^{\mu_1}\xi_{\mu_1}\right]\\ & -\frac{1}{2}\lambda_{\mu_3}\left[\big(\nabla_{\mu_2}\xi_{\mu_4}\big)\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1}-\frac{3R}{D(D-1)}\right)+\frac{1}{2}\xi_{\mu_4}\xi^{\mu_1}\nabla_{\mu_2}\xi_{\mu_1}\right]\\ & +\frac{R}{D(D-1)}\lambda_{\mu_4}\left[g_{\mu_2\mu_3}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1}-\frac{R}{D(D-1)}\right)+\nabla_{\mu_3}\xi_{\mu_2}+\nabla_{\mu_2}\xi_{\mu_3}\right]\\ & -\frac{1}{2}\lambda_{\mu_4}\left[\xi^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_1}+\big(\nabla_{\mu_2}\xi^{\mu_1}\big)\nabla_{\mu_3}\xi_{\mu_1}\right].\end{aligned} \tag{A.77}$$

Now, let us find this result from the recursion relations. The $E^{(k,n,1)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}}$ terms that we need to calculate are

$$\lambda^{\mu_1}\left(\prod_{i=1}^{3}\nabla_{\mu_i}\right)\xi_{\mu_4} = \sum_{k=2}^{4}\lambda_{\mu_k}E^{(k,4,1)}_{\mu_{r_1}\mu_{r_2}}$$
$$\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_4} = \lambda_{\mu_2}E^{(2,4,1)}_{\mu_3\mu_4} + \lambda_{\mu_3}E^{(3,4,1)}_{\mu_2\mu_4} + \lambda_{\mu_4}E^{(4,4,1)}_{\mu_2\mu_3}. \tag{A.78}$$

Using the recursion relations (A.54)–(A.56), one has

$$\begin{aligned} E^{(2,4,1)}_{\mu_3\mu_4} &= \frac{1}{2}\left[\sum_{k=3}^{4}\xi_{\mu_k}E^{(k-1,3,1)}_{\mu_{m_1}} - \xi^{\mu_1}\nabla_{\mu_1}\left(\prod_{i=3}^{3}\nabla_{\mu_i}\right)\xi_{\mu_4}\right] \\ &= \frac{1}{2}\left[\xi_{\mu_3}E^{(2,3,1)}_{\mu_4} + \xi_{\mu_4}E^{(3,3,1)}_{\mu_3} - \xi^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_3}\xi_{\mu_4}\right], \end{aligned} \tag{A.79}$$

$$\begin{aligned} E^{(3,4,1)}_{\mu_2\mu_4} &= \frac{R}{D(D-1)}\left(\prod_{i_1=3}^{2}\nabla_{\mu_{i_1}}\right)\nabla_{\mu_2}\left(\prod_{i_2=4}^{3}\nabla_{\mu_{i_2}}\right)\xi_{\mu_4} \\ &\quad -\frac{R}{D(D-1)}\sum_{\substack{s=3\\(s\neq 3)}}^{4} g_{\mu_2\mu_s}E^{(1,2,s-2)} + \nabla_{\mu_2}E^{(2,3,1)}_{\mu_4} \\ &= \frac{R}{D(D-1)}\left(\nabla_{\mu_2}\xi_{\mu_4} - g_{\mu_2\mu_4}E^{(1,2,2)}\right) + \nabla_{\mu_2}E^{(2,3,1)}_{\mu_4}, \end{aligned} \tag{A.80}$$

$$\begin{aligned} E^{(4,4,1)}_{\mu_2\mu_3} &= \frac{R}{D(D-1)}\left[\left(\prod_{i=3}^{3}\nabla_{\mu_i}\right)\xi_{\mu_2} - \sum_{s=3}^{3} g_{\mu_2\mu_s}E^{(2,2,s-2)}\right] + \nabla_{\mu_2}E^{(3,3,1)}_{\mu_3} \\ &= \frac{R}{D(D-1)}(\nabla_{\mu_3}\xi_{\mu_2} - g_{\mu_2\mu_3}E^{(2,2,1)}) + \nabla_{\mu_2}E^{(3,3,1)}_{\mu_3}. \end{aligned} \tag{A.81}$$

We have already calculated $E^{(1,2,2)}$ and $E^{(2,2,1)}$. The term $E^{(2,3,1)}_{\mu_4}$ is the coefficient of λ_{μ_2} in $\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}$ which is

$$E^{(2,3,1)}_{\mu_4} = -\frac{1}{2}\left[\xi_{\mu_4}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{R}{D(D-1)}\right) + \xi^{\mu_1}\nabla_{\mu_1}\xi_{\mu_4}\right]. \tag{A.82}$$

The term $E^{(3,3,1)}_{\mu_3}$ is the coefficient of λ_{μ_3} in, again, $\lambda^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_2}\xi_{\mu_3}$ which is

$$E^{(3,3,1)}_{\mu_3} = -\frac{1}{2}\xi^{\mu_1}\nabla_{\mu_3}\xi_{\mu_1} + \frac{R}{D(D-1)}\xi_{\mu_3}. \tag{A.83}$$

Putting these results in $E^{(2,4,1)}_{\mu_3\mu_4}$, $E^{(3,4,1)}_{\mu_2\mu_4}$, and $E^{(4,4,1)}_{\mu_2\mu_3}$ yields

$$E^{(2,4,1)}_{\mu_3\mu_4} = -\frac{1}{4}\xi_{\mu_3}\xi_{\mu_4}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{3R}{D(D-1)}\right) - \frac{1}{2}\left(\xi^{\mu_1}\nabla_{\mu_1}\nabla_{\mu_3}\xi_{\mu_4} + \frac{1}{2}\xi_{\mu_4}\xi^{\mu_1}\nabla_{\mu_3}\xi_{\mu_1} + \frac{1}{2}\xi_{\mu_3}\xi^{\mu_1}\nabla_{\mu_1}\xi_{\mu_4}\right), \tag{A.84}$$

$$E^{(3,4,1)}_{\mu_2\mu_4} = -\frac{1}{2}\left[\left(\nabla_{\mu_2}\xi_{\mu_4}\right)\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{3R}{D(D-1)}\right) + \frac{1}{2}\xi_{\mu_4}\xi^{\mu_1}\nabla_{\mu_2}\xi_{\mu_1}\right] - \frac{1}{2}\left[\xi^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_1}\xi_{\mu_4} + \left(\nabla_{\mu_2}\xi^{\mu_1}\right)\nabla_{\mu_1}\xi_{\mu_4} - \frac{R}{D(D-1)}g_{\mu_2\mu_4}\xi^{\mu_1}\xi_{\mu_1}\right], \tag{A.85}$$

$$E^{(4,4,1)}_{\mu_2\mu_3} = \frac{R}{D(D-1)}\left[\nabla_{\mu_2}\xi_{\mu_3} + \nabla_{\mu_3}\xi_{\mu_2} + g_{\mu_2\mu_3}\left(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{R}{D(D-1)}\right)\right] - \frac{1}{2}\left[\xi^{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_1} + (\nabla_{\mu_2}\xi^{\mu_1})\nabla_{\mu_3}\xi_{\mu_1}\right], \tag{A.86}$$

which are the same as the ones that can be obtained from (A.77). After these lemmas and examples, we now have the proper arsenal to prove the theorem.

Proof of theorem 2. As a result of the previous lemmas, we showed that $\lambda^{\mu_j}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}$ is λ-reducible as

$$\lambda^{\mu_j}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \sum_{\substack{k=1\\(k\neq j)}}^{n}\lambda_{\mu_k}E^{(k,n,j)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}},$$

where the $(0, n-2)$ rank tensors $E^{(k,n,j)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}}$ are related to the lower orders with the recursion relations (A.54)–(A.56) for the $j = 1$ contraction pattern and with the recursion relations (A.39) and (A.40) for the contraction patterns of $1 < j \leqslant n$. From these recursion relations, one can see that the $E^{(k,n,j)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}}$ tensors are built from the structures

$$\xi_{\mu_k}, \qquad E^{(k,n-1,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}, \qquad \left(\prod_{i=1}^{n-2}\nabla_{\mu_i}\right)\xi_{\mu_{n-1}}, \qquad \nabla_{\mu_1}E^{(k,n-1,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}, \tag{A.87}$$

$$g_{\mu_2\mu_s}, \qquad E^{(k,n-2,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-4}}}, \qquad \left(\prod_{i=1}^{n-3}\nabla_{\mu_i}\right)\xi_{\mu_{n-2}}, \tag{A.88}$$

where the building blocks of $E^{(k,n-1,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-3}}}$ and $E^{(k,n-2,j)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-4}}}$ are

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r}, \qquad r = 2, 3, \ldots, n-2, \tag{A.89}$$

and

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r}, \qquad r=2,3,\ldots,n-3, \tag{A.90}$$

respectively. Therefore, the building blocks of the $E^{(k,n,j)}_{\mu_{r_1}\mu_{r_2}\ldots\mu_{r_{n-2}}}$ tensors are

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r}, \qquad r=2,3,\ldots,n-1. \tag{A.91}$$

Note that contracting the $(0,n)$ rank tensor $\nabla^{n-1}\xi$ with λ reduces the derivative order such that the highest derivative order term now becomes $\nabla^{n-2}\xi$.

Now, let us discuss the contractions of $\nabla^{n-1}\xi$ with more than one λ. In this regard, the important thing that should be noticed in the one-λ contraction result is that the building blocks of the tensor structures produced by the one-λ contraction of $\nabla^{n-1}\xi$, which are the metric, ξ, and the lower order derivatives of ξ, are all λ-reducible under one-λ contraction. For a further λ contraction, when the λ vector is contracted with the derivatives of ξ, again the lower order derivatives of ξ will appear together with the metric and ξ as building blocks. Due to continuous appearance of the same λ-reducible building blocks, $\nabla^{n-1}\xi$ should be λ-conserving.

To be more explicit, let us first consider the contraction of the $(0,n)$ rank tensor $\nabla^{n-1}\xi$ with two λ vectors; that is

$$\lambda^{\mu_{j_1}}\lambda^{\mu_{j_2}}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}, \tag{A.92}$$

where $\{j_1,j_2\}$ is a subset of $\{1,2,\ldots n\}$. Using the one-λ contraction result (A.29), one has

$$\lambda^{\mu_{j_1}}\lambda^{\mu_{j_2}}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} = \sum_{\substack{k_1=1\\(k_1\neq j_1,j_2)}}^{n} \lambda_{\mu_{k_1}}\lambda^{\mu_{j_2}}E^{(k_1,n,j_1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-2}}}, \tag{A.93}$$

where we know that the $E^{(k_1,n,j_1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-2}}}$ tensors are the rank $(0,n-2)$ contractions of the building blocks

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r}, \qquad r=2,3,\ldots,n-1. \tag{A.94}$$

Since the one-λ contraction of these building blocks are λ-reducible, the $E^{(k_1,n,j_1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-2}}}$ tensors should also be λ-reducible as

$$\lambda^{\mu_{j_2}}E^{(k_1,n,j_1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-2}}} = \sum_{\substack{k_2=1\\(k_2\neq k_1,j_1,j_2)}}^{n} \lambda_{\mu_{k_2}}E^{(k_1,k_2,n,j_1,j_2)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-4}}}, \tag{A.95}$$

where $(r_1,r_2,\ldots,r_{n-4})$ is an *increasing* sequence constructed from $\{1,2,\ldots,n\}\setminus\{j_1,k_1,j_2,k_2\}$. We know that contracting $\nabla^{n-1}\xi$ with one λ reduces the highest derivative order to $\nabla^{n-2}\xi$. Then, the highest derivative order for the building blocks of $E^{(k_1,k_2,n,j_1,j_2)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-4}}}$ should be one order less than the highest derivative order for the building blocks of $E^{(k_1,n,j_1)}_{\mu_{m_1}\mu_{m_2}\cdots\mu_{m_{n-2}}}$ given in (A.94).

Thus, the $E^{(k_1,k_2,n,j_1,j_2)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-4}}}$ tensors are the rank $(0,n-4)$ contractions of the building blocks

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r}, \qquad r=2,3,\ldots,n-2. \tag{A.96}$$

The final form of the two-λ contraction of $\nabla^{n-1}\xi$,

$$\lambda^{\mu_{j_1}}\lambda^{\mu_{j_2}}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}, \tag{A.97}$$

becomes

$$\begin{aligned}\lambda^{\mu_{j_1}}\lambda^{\mu_{j_2}}\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n} &= \sum_{\substack{k_1=1\\(k_1\neq j_1,j_2)}}^{n}\lambda_{\mu_{k_1}}\sum_{\substack{k_2=1\\(k_2\neq k_1,j_1,j_2)}}^{n}\lambda_{\mu_{k_2}}E^{(k_1,k_2,n,j_1,j_2)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-4}}}\\ &= \sum_{\substack{k_1=1\\(k_1\neq j_1,j_2)}}^{n}\ \sum_{\substack{k_2=1\\(k_2\neq k_1,j_1,j_2)}}^{n}\lambda_{\mu_{k_1}}\lambda_{\mu_{k_2}}E^{(k_1,k_2,n,j_1,j_2)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-4}}}.\end{aligned} \tag{A.98}$$

Here, notice the pattern that the two-λ contraction of $\nabla^{n-1}\xi$ becomes a sum of $(0,n-2)$ tensors which are decomposed into two λ vectors and rank $(0,n-4)$ tensors $E^{(k_1,k_2,n,j_1,j_2)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-4}}}$ while the one-λ contraction of $\nabla^{n-1}\xi$ becomes a sum of $(0,n-1)$ tensors which are decomposed into one λ vector and rank $(0,n-2)$ tensors $E^{(k_1,n,j_1)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2}}}$.

Further λ contractions of $\nabla^{n-1}\xi$ also have the same pattern: contracting the rank $(0,n)$ tensor $\nabla^{n-1}\xi$ with p number of λ tensors yields a sum of $(0,n-p)$ rank tensors which can be decomposed into p number of λ tensors and rank $(0,n-2p)$ tensors

$$E^{(k_1,k_2,\ldots,k_p,n,j_1,j_2,\ldots j_p)}_{\mu_{r_1}\mu_{r_2}\cdots\mu_{r_{n-2p}}}, \tag{A.99}$$

for which the building blocks are

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r}, \qquad r=2,3,\ldots,n-p. \tag{A.100}$$

More explicitly, the contraction of $\nabla^{n-1}\xi$ with p number of λ vectors can be represented as

$$\left(\prod_{r=1}^{p}\lambda^{\mu_{j_r}}\right)\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}, \tag{A.101}$$

where $\{j_1,j_2,\ldots,j_p\}$ is a subset of $\{1,2,\ldots,n\}$, and following the pattern we developed, this term becomes[14]

[14] Assuming n is sufficiently larger than p.

$$\left(\prod_{s=1}^{p}\lambda^{\mu_{j_s}}\right)\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}=\prod_{s=1}^{p}\left(\sum_{\substack{k_s=1\\(k_s\neq j_1,\dots,j_p,k_1,\dots,k_{s-1})}}^{n}\lambda_{\mu_{k_s}}\right)E^{(k_1,k_2,\dots,k_p,n,j_1,j_2,\dots,j_p)}_{\mu_{r_1}\mu_{r_2}\dots\mu_{r_{n-2p}}},\tag{A.102}$$

where $\{r_1,r_2,\dots,r_{n-2p}\}$ is an increasing sequence constructed from $\{1,2,\dots,n\}\setminus\{j_i,k_i:\ 1\leqslant k\leqslant p\}$. This result shows that the maximum number of λ contractions with $\nabla^{n-1}\xi$ before getting a zero is $n/2$ for even n and $(n-1)/2$ for odd n, and one gets the following results, respectively;

$$\left(\prod_{s=1}^{\frac{n}{2}}\lambda^{\mu_{j_s}}\right)\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}=\prod_{s=1}^{\frac{n}{2}}\left(\sum_{\substack{k_s=1\\(k_s\neq j_1,\dots,j_{n/2},k_1,\dots,k_{s-1})}}^{n}\lambda_{\mu_{k_s}}\right)E^{(k_1,k_2,\dots,k_{n/2},n,j_1,j_2,\dots,j_{n/2})},\tag{A.103}$$

where the building blocks for $E^{(k_1,k_2,\dots,k_{n/2},n,j_1,j_2,\dots,j_{n/2})}$ are

$$\xi_{\mu_1},\qquad\left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r},\qquad r=2,3,\dots,\frac{n}{2},\tag{A.104}$$

and

$$\left(\prod_{s=1}^{\frac{n-1}{2}}\lambda^{\mu_{j_s}}\right)\left(\prod_{i=1}^{n-1}\nabla_{\mu_i}\right)\xi_{\mu_n}=\prod_{s=1}^{\frac{n-1}{2}}\left(\sum_{\substack{k_s=1\\(k_s\neq j_1,\dots,j_{(n-1)/2},k_1,\dots,k_{s-1})}}^{n}\lambda_{\mu_{k_s}}\right)E^{(k_1,k_2,\dots,k_{(n-1)/2},n,j_1,j_2,\dots,j_{(n-1)/2})}_{\mu_{m_1}},\tag{A.105}$$

where $m_1\in\{1,2,\dots,n\}\setminus\{j_i,k_i:\ 1\leqslant i\leqslant(n-1)/2\}$ and the building blocks for $E^{(k_1,k_2,\dots,k_{(n-1)/2},n,j_1,j_2,\dots,j_{(n-1)/2})}_{\mu_{m_1}}$ are

$$\xi_{\mu_1},\qquad\left(\prod_{i=1}^{r-1}\nabla_{\mu_i}\right)\xi_{\mu_r},\qquad r=2,3,\dots,\frac{n+1}{2}.\tag{A.106}$$

To conclude, the $(0,n)$ rank tensor $\nabla^{n-1}\xi$ is λ-conserving since with each λ contraction, the λ-weight of the resulting tensor structure increases by one. This proves the theorem. □

Example 5. For odd n, let us consider $n=3$ case, that is $\nabla\nabla\xi$, for which one λ contractions that we found in (A.21), (A.22) and (A.24) are the last nonzero terms. It can be verified immediately that a further λ contraction with any of the one-λ contraction patterns of $\nabla\nabla\xi$ given in (A.21), (A.22) and (A.24) yields a zero. For the even n case, let us consider $\nabla^3\xi$ for which two λ contraction is the last nonzero order. As an example, let us study a further λ contraction of the $j_1=1$ contraction pattern of $\nabla^3\xi$ given in (A.77), so after a long calculation $\lambda^{\mu_1}\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_4}$ reduces to

$$\lambda^{\mu_1}\lambda^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\xi_{\mu_4} = \frac{1}{2}\lambda_{\mu_3}\lambda_{\mu_4}\xi^{\mu_1}\xi^{\mu_2}\nabla_{\mu_1}\xi_{\mu_2} - \frac{R}{D(D-1)}\lambda_{\mu_3}\lambda_{\mu_4}\left(\frac{3}{4}\xi^{\mu_2}\xi_{\mu_2} - \frac{R}{D(D-1)}\right), \tag{A.107}$$

which is, as expected, the last nonzero order constructed from the building blocks ξ_μ and $\nabla_\mu\xi_\nu$.

Theorem 3. *The rank* $(0, n)$ *tensor* $\nabla^n V$ *is* λ*-conserving.*

Proof. The proof follows the same lines as the proof of theorem 2. For the first step of the induction part of the proof, now one has he equations $\lambda^\mu\partial_\mu V = 0$ and

$$\lambda^\mu\nabla_\mu\partial_\nu V = \lambda^\mu\nabla_\nu\partial_\mu V = -\frac{1}{2}\lambda_\nu\xi^\mu\partial_\mu V. \tag{A.108}$$

In addition, the tensorial structures now involve the covariant derivatives of V in addition to ξ_μ and its covariant derivatives. Since the proof involves the same cumbersome steps without new ideas, we do not display it here. □

Now, we can give the proof of the main theorem in the text (theorem 1) based on above results:

Proof of theorem 1. Remember that $\mathcal{E}_{\mu_1\dots\mu_s}$ represents the sum of rank $(0, s)$ tensors which can be decomposed into $2(n_0 + m)$ number of λ vectors and rank $(0, s - 2n_0 - 2m)$ tensor structures which are obtained from the contractions of the following building blocks

$$g_{\mu_1\mu_2}, \qquad \xi_{\mu_1}, \qquad \left(\prod_{i=1}^{r}\nabla_{\mu_i}\right)\xi_{\mu_{r+1}}, \qquad \left(\prod_{i=1}^{r+2}\nabla_{\mu_i}\right)V, \qquad r = 1, 2, \dots, n_m, \tag{A.109}$$

which are all λ-conserving as we have shown. Then, to have a nonzero $E_{\mu\nu}$ two-tensor out of $\mathcal{E}_{\mu_1\dots\mu_s}$, one must have at most two λ vectors in $\mathcal{E}_{\mu_1\dots\mu_s}$. If there is more than two λ vectors in $\mathcal{E}_{\mu_1\dots\mu_s}$, then they eventually yield a zero contraction since any nonzero contraction for $\mathcal{E}_{\mu_1\dots\mu_s}$ conserves the number of the λ vectors. Thus, one should start with $R_{\alpha\beta\rho\sigma}$ or $\nabla_{\mu_1}\nabla_{\mu_2}\dots\nabla_{\mu_m}R_{\alpha\beta\rho\sigma}$ to have nonzero $E_{\mu\nu}$ two-tensors. The remaining part of the proof on the structure of the nonzero $E_{\mu\nu}$ tensors follows as given in section 3. □

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