

MATH 116 Second Midterm Exam, Solutions

1. a) Evaluate $\iint_R x \, dA$ as the iterated integral $\int (\int x \, dx) dy$, where R is the region inside the circle $x^2 + y^2 = 9$ and over the parabola $y = x^2 + 4x + 3$.

Solution: If $y = x^2 + 4x + 3$, then $x = -2 \pm \sqrt{y+1}$. Therefore,

$$\begin{aligned} & \int_{-1}^0 \int_{-2-\sqrt{y+1}}^{-2+\sqrt{y+1}} x \, dx \, dy + \int_0^3 \int_{-\sqrt{9-y^2}}^{-2+\sqrt{y+1}} x \, dx \, dy = \\ & \int_{-1}^0 \frac{1}{2} [(-2+\sqrt{y+1})^2 - (-2-\sqrt{y+1})^2] \, dy + \int_0^3 \frac{1}{2} [(-2+\sqrt{y+1})^2 - (-\sqrt{9-y^2})^2] \, dy = \\ & -4 \int_{-1}^0 \sqrt{y+1} \, dy + \frac{1}{2} \int_0^3 (4 - 4\sqrt{y+1} + y + 1 - 9 + y^2) \, dy = -4 \int_0^1 \sqrt{t} \, dt + \\ & \frac{1}{2} \int_0^3 (y^2 + y - 4) \, dy - 2 \int_0^3 \sqrt{y+1} \, dy = -4 \cdot \frac{2}{3} t^{3/2} \Big|_0^1 + \frac{1}{2} \left(\frac{27}{3} + \frac{9}{2} - 4 \cdot 3 \right) - \\ & 2 \int_1^4 \sqrt{t} \, dt = -\frac{8}{3} + \frac{9}{2} + \frac{9}{4} - 6 - 2 \cdot \frac{2}{3} (4^{3/2} - 1) = -\frac{8}{3} + \frac{27}{4} - 6 - \frac{28}{3} = -11 \frac{1}{4}. \end{aligned}$$

1. b) Evaluate $\int_0^4 \int_{x/2}^2 e^{y^2} \, dy \, dx$.

Solution: We change the order of integration:

$$\int_0^2 \int_0^{2y} e^{y^2} \, dx \, dy = \int_0^2 e^{y^2} 2y \, dy = e^{y^2} \Big|_0^2 = e^4 - 1.$$

2. Evaluate $\iint_R \frac{x^2 \sin[\pi(x^2 + y^2)]}{x^2 + y^2} \, dA$ over the region R that lies between the circumferences $x^2 + y^2 = 1/4$, $x^2 + y^2 = 1$ and above the line $y = x$.

Solution: In polar coordinates we get

$$\begin{aligned} & \int_{\pi/4}^{5\pi/4} \int_{1/2}^1 \frac{r^2 \cos^2 \theta}{r^2} \sin(\pi r^2) r \, dr \, d\theta = \int_{\pi/4}^{5\pi/4} \frac{1 + \cos 2\theta}{2} \, d\theta \cdot \int_{1/2}^1 r \sin(\pi r^2) \, dr = \\ & \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \Big|_{\pi/4}^{5\pi/4} \cdot \left(\frac{-\cos(\pi r^2)}{2\pi} \right) \Big|_{1/2}^1 = \left[\frac{\pi}{2} + \frac{1}{4} \left(\sin \frac{5\pi}{2} - \sin \frac{\pi}{2} \right) \right] \cdot \frac{1}{2\pi} \left[-\cos \pi + \cos \frac{\pi}{4} \right] \\ & = \left[\frac{\pi}{2} + \frac{1}{4} \cdot (1 - 1) \right] \cdot \frac{1}{2\pi} \left[1 + \frac{\sqrt{2}}{2} \right] = \frac{1}{4} \cdot \frac{2 + \sqrt{2}}{2} = \frac{2 + \sqrt{2}}{8}. \end{aligned}$$

3. a) Consider the integral $\iiint_D f(x, y, z) \, dV$, where D is the region bounded above by the hemisphere $z = \sqrt{9 - x^2 - y^2}$, below by the xy -plane and on sides by the cylinder $x^2 + y^2 = 1$. Write the integral in cylindrical coordinates **and** in

spherical coordinates.

Solution:

$$\begin{aligned} \int \int \int_D f(x, y, z) dV &= \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{9-r^2}} f(r \cos\theta, r \sin\theta, z) r dz dr d\theta = \\ &= \int_0^{2\pi} \int_0^{\arcsin 1/3} \int_0^3 f(\rho \cos\theta \sin\varphi, \rho \sin\theta \sin\varphi, \rho \cos\varphi) \rho^2 \sin\varphi d\rho d\varphi d\theta + \\ &= \int_0^{2\pi} \int_{\arcsin 1/3}^{\pi/2} \int_0^{1/\sin\varphi} f(\rho \cos\theta \sin\varphi, \rho \sin\theta \sin\varphi, \rho \cos\varphi) \rho^2 \sin\varphi d\rho d\varphi d\theta. \end{aligned}$$

For spherical coordinates we use the following calculations:

$$\rho = 3, r = 1 \Rightarrow \sin\varphi_0 = 1/3. \text{ Also, } x^2 + y^2 = 1 \Rightarrow \rho \sin\varphi = 1.$$

3. b) Find the volume of the region enclosed by the surfaces $z = x^2 + y^2$ and $z = (x^2 + y^2 + 1)/2$.

Solution: In cylindrical coordinates the surfaces $z = r^2$ and $z = (r^2 + 1)/2$ intersect when $r = 1$. Therefore,

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \int_{r^2}^{(r^2+1)/2} r dz dr d\theta = 2\pi \int_0^1 r \left(\frac{r^2+1}{2} - r^2 \right) dr = \pi \int_0^1 r(1-r^2) dr = \\ &= \pi(1/2 - 1/4) = \pi/4. \end{aligned}$$

4. Let D be the region in the first quadrant bounded by the hyperbolas $xy = 1, xy = 9$ and the lines $y = x, y = 4x$. Evaluate $\int \int_D (\sqrt{y/x} + \sqrt{xy}) dA$.

Solution: We use the substitution $u = xy$ and $v = y/x$. Then in new coordinates u, v the region D is the rectangle $1 \leq u \leq 9, 1 \leq v \leq 4$. Here,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = y/x + y/x = 2v. \text{ Therefore, } \frac{\partial(x, y)}{\partial(u, v)} = (2v)^{-1} \text{ and}$$

$$\begin{aligned} \int \int_D (\sqrt{y/x} + \sqrt{xy}) dA &= \int_1^9 \int_1^4 \frac{\sqrt{v} + \sqrt{u}}{2v} dv du = \int_1^9 \left(\sqrt{v} + \frac{1}{2} \sqrt{u} \ln v \right) \Big|_{v=1}^{v=4} du = \\ &= \int_1^9 \left(1 + \frac{\ln 4 - \ln 1}{2} \sqrt{u} \right) du = (u + \ln 2 \cdot \frac{2}{3} u^{3/2}) \Big|_1^9 = 8 + \ln 2 \cdot \frac{2}{3} \cdot 26 = 8 + \frac{52}{3} \ln 2. \end{aligned}$$

5. The region R is bounded by the planes $4x + y - 2z = 0, x - y + 2z = 0, y = 0$, and $x - y - 3z + 5 = 0$. Use the transformation $x = u - 2w, y = 2v, z = 2u + v + w$ to evaluate $\int \int \int_R y dV$.

Solution: Here, $0 = 4x + y - 2z = 4u - 8w + 2v - 4u - 2v - 2w = -10w$, $0 = x - y + 2z = u - 2w - 2v + 4u + 2v + 2w = 5u$, $0 = y = 2v$, and $0 = x - y - 3z + 5 = u - 2w - 2v - 6u - 3v - 3w + 5 = 5(1 - u - v - w)$. Therefore

the preimage of R under the given transformation is the simplex G defined by the conditions $u \geq 0, v \geq 0, w \geq 0, u + v + w \leq 1$. The Jacobian of transformation is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 10.$$

Thus,

$$\begin{aligned} \int \int \int_R y \, dV &= \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 2v \cdot 10 \, dw \, dv \, du = 20 \int_0^1 \int_0^{1-u} v(1-u-v) \, dv \, du = \\ &20 \int_0^1 [(1-u)v^2/2 - v^3/3] \Big|_{v=0}^{v=1-u} \, du = 20 \int_0^1 (1-u)^3 (1/2 - 1/3) \, du = \frac{20}{6} \int_0^1 t^3 \, dt = \\ &10/3 \cdot 1/4 = 5/6. \end{aligned}$$