MATH 116 Second Midterm Exam, Solutions

1. a) Evaluate $\int\int_R x\,dA$ as the iterated integral $\int(\int x\,dx)dy$, where R is the region inside the circle $x^2+y^2=9$ and over the parabola $y=x^2+4x+3$.

Solution: If $y=x^2+4x+3$, then $x=-2\pm\sqrt{y+1}$. Therefore,

$$\int_{-1}^{0}\int_{-2-\sqrt{y+1}}^{-2+\sqrt{y+1}} x\,dx\,dy+\int_{0}^{3}\int_{-\sqrt{9-y^2}}^{-2+\sqrt{y+1}} x\,dx\,dy=$$

$$\int_{-1}^{0}\frac{1}{2}[(-2+\sqrt{y+1})^2-(-2-\sqrt{y+1})^2]\,dy+\int_{0}^{3}\frac{1}{2}[(-2+\sqrt{y+1})^2-(-\sqrt{9-y^2})^2]\,dy=$$

$$-4\int_{-1}^{0}\sqrt{y+1}\,dy+\frac{1}{2}\int_0^3(4-4\sqrt{y+1}+y+1-9+y^2)\,dy=-4\int_0^1\sqrt{t}\,dt+$$

$$\frac{1}{2}\int_0^3(y^2+y-4)\,dy-2\int_0^3\sqrt{y+1}\,dy=-4\cdot 2/3\,t^{3/2}\,|_0^1+\frac{1}{2}\left(\frac{27}{3}+\frac{9}{2}-4\cdot 3\right)-$$

$$2\int_1^4\sqrt{t}\,dt=-\frac{8}{3}+\frac{9}{2}+\frac{9}{4}-6-2\cdot\frac{2}{3}(4^{3/2}-1)=-\frac{8}{3}+\frac{27}{4}-6-\frac{28}{3}=-11\frac{1}{4}.$$

1. b) Evaluate $\int_0^4\int_{x/2}^2 e^{y^2}\,dydx.$

Solution: We change the order of integration:

$$\int_0^2\int_0^{2y} e^{y^2}\,dx\,dy=\int_0^2 e^{y^2}\,2y\;dy=e^{y^2}\,|_0^2=e^4-1.$$

2. Evaluate $\displaystyle\int\int_R\frac{x^2\,sin[\pi(x^2+y^2)]}{x^2+y^2}\,dA$ over the region R that lies between the circumferences $x^2+y^2=1/4$, $x^2+y^2=1$ and above the line $y=x$.

Solution: In polar coordinates we get

$$\int_{\pi/4}^{5\pi/4}\int_{1/2}^{1}\frac{r^2\,cos^2\theta}{r^2}\,sin(\pi\,r^2)\,r\,dr\,d\theta=\int_{\pi/4}^{5\pi/4}\frac{1+cos2\theta}{2}\,d\theta\cdot\int_{1/2}^{1}r\,sin(\pi\,r^2)\,dr=$$

$$\left(\frac{\theta}{2}+\frac{sin2\,\theta}{4}\right)|_{\pi/4}^{5\pi/4}\cdot\left(\frac{-cos(\pi\,r^2)}{2\pi}\right)|_{1/2}^{1}=\left[\frac{\pi}{2}+\frac{1}{4}\left(sin\frac{5\pi}{2}-sin\frac{\pi}{2}\right)\right]\cdot\frac{1}{2\pi}\left[-cos\pi+cos\frac{\pi}{4}\right]$$

$$=\left[\frac{\pi}{2}+\frac{1}{4}\cdot(1-1)\right]\cdot\frac{1}{2\pi}\left[1+\frac{\sqrt{2}}{2}\right]=\frac{1}{4}\cdot\frac{2+\sqrt{2}}{2}=\frac{2+\sqrt{2}}{8}.$$

3. a) Consider the integral $\int\int\int_D f(x,y,z)\,dV$, where D is the region bounded above by the hemisphere $z=\sqrt{9-x^2-y^2}$, below by the $xy-$ plane and on sides by the cylinder $x^2+y^2=1$. Write the integral in cylindrical coordinates **and** in spherical coordinates.

Solution:

$$\int\int\int_D f(x,y,z)\,dV = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{9-r^2}} f(r\,cos\theta, r\,sin\theta, z)\,r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^{arcsin1/3}\int_0^3 f(\rho\,cos\theta\,sin\varphi, \rho\,sin\theta\,sin\varphi, \rho\,cos\varphi)\,\rho^2\,sin\varphi\,d\rho\,d\varphi\,d\theta +$$

$$\int_0^{2\pi}\int_{arcsin1/3}^{\pi/2}\int_0^{1/sin\varphi} f(\rho\,cos\theta\,sin\varphi, \rho\,sin\theta\,sin\varphi, \rho\,cos\varphi)\,\rho^2\,sin\varphi\,d\rho\,d\varphi\,d\theta.$$

For spherical coordinates we use the following calculations:

$$\rho = 3,\ r = 1 \Rightarrow sin\varphi_0 = 1/3.\ \text{ Also, } x^2+y^2 = 1 \Rightarrow \rho\,sin\varphi = 1.$$

3. b) Find the volume of the region enclosed by the surfaces $z = x^2 + y^2$ and $z = (x^2+y^2+1)/2$.

Solution: In cylindrical coordinates the surfaces $z = r^2$ and $z = (r^2+1)/2$ intersect when $r = 1$. Therefore,

$$V = \int_0^{2\pi}\int_0^1\int_{r^2}^{r^2+1} r\,dz\,dr\,d\theta = 2\pi\int_0^1 r\left(\frac{r^2+1}{2} - r^2\right)dr = \pi\int_0^1 r(1-r^2)\,dr =$$

$\pi(1/2 - 1/4) = \pi/4.$

4. Let D be the region in the first quadrant bounded by the hyperbolas $xy = 1, xy = 9$ and the lines $y = x, y = 4x$. Evaluate $\displaystyle\int\int_D (\sqrt{y/x} + \sqrt{xy}\,)\,dA.$

Solution: We use the substitution $u = xy$ and $v = y/x$. Then in new coordinates u, w the region D is the rectangle $1 \le u \le 9, 1 \le v \le 4$. Here, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = y/x + y/x = 2v$. Therefore, $\frac{\partial(x,y)}{\partial(u,v)} = (2v)^{-1}$ and

$$\int\int_D (\sqrt{y/x}+\sqrt{xy}\,)\,dA = \int_1^9\int_1^4 \frac{\sqrt{v}+\sqrt{u}}{2v}\,dv\,du = \int_1^9 \left(\sqrt{v} + \frac{1}{2}\sqrt{u}\,lnv\right)\Big|_{v=1}^{v=4}\,du =$$

$$\int_1^9 \left(1 + \frac{ln4 - ln1}{2}\sqrt{u}\right)du = (u + ln2\cdot\frac{2}{3}u^{3/2})\,\Big|_1^9 = 8 + ln2\cdot\frac{2}{3}\cdot 26 = 8 + \frac{52}{3}\,ln2.$$

5. The region R is bounded by the planes $4x + y - 2z = 0, x - y + 2z = 0, y = 0,$ and $x - y - 3z + 5 = 0$. Use the transformation $x = u - 2w, y = 2v, z = 2u + v + w$ to evaluate $\int\int\int_R y\,dV$.

Solution: Here, $0 = 4x + y - 2z = 4u - 8w + 2v - 4u - 2v - 2w = -10w$, $0 = x - y + 2z = u - 2w - 2v + 4u + 2v + 2w = 5u$, $0 = y = 2v$, and $0 = x - y - 3z + 5 = u - 2w - 2v - 6u - 3v - 3w + 5 = 5(1 - u - v - w)$. Therefore the preimage of R under the given transformation is the simplex G defined by the conditions $u \geq 0, v \geq 0, w \geq 0, u+v+w \leq 1$. The Jacobian of transformation is

$$\frac{\partial(x,\, y,\, z)}{\partial(u,\, v,\, w)} = \begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 10.$$

Thus,

$$\int\!\!\int\!\!\int_R y\, dV = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 2v \cdot 10\, dw\, dv\, du = 20 \int_0^1 \int_0^{1-u} v(1-u-v) dv\, du =$$

$$20 \int_0^1 [(1-u)\, v^2/2 - v^3/3]\,\Big|_{v=0}^{v=1-u}\, du = 20 \int_0^1 (1-u)^3\, (1/2 - 1/3)\, du = \frac{20}{6} \int_0^1 t^3\, dt =$$

$10/3 \cdot 1/4 = 5/6.$