Special relativity is reformulated as a symmetry property of space-time: space-time exchange invariance. The additional hypothesis of spatial homogeneity is then sufficient to derive the Lorentz transformation without reference to the traditional form of the Principle of Special Relativity. The kinematical version of the latter is shown to be a consequence of the Lorentz transformation. As a dynamical application, the laws of electrodynamics and magnetodynamics are derived from those of electrostatics and magnetostatics respectively. The four-vector nature of the electromagnetic potential plays a crucial role in the last two derivations. © 2001 American Association of Physics Teachers. [DOI: 10.1119/1.1344165]
multiplied by a universal parameter $V$ with the dimensions of velocity. The new time coordinate with dimension $[L]$, $x^0 = Vt$, (1.1)

may be called the "causality radius" to distinguish it from the Cartesian spatial coordinate $x$ or the invariant interval $s$. Since space is three dimensional and time is one dimensional, there is a certain ambiguity in the definition of the exchange operation in (I). Depending on the case under discussion, the space coordinate may be either the magnitude of the spatial vector $x = |\vec{x}|$, or a Cartesian component $x^1$, $x^2$, $x^3$. For any physical problem with a preferred spatial direction (which is the case for the LT), then, by a suitable choice of coordinate system, the identification $x = x^1$, $x^2 = x^3 = 0$ is always possible. The exchange operation in (I) is then simply $x^0 \leftrightarrow x^1$. Formally, the exchange operation is defined by the equations

$$STEx^0 = x^1, \quad (1.2)$$

$$STEx^1 = x^0, \quad (1.3)$$

$$(STE)^2 = 1, \quad (1.4)$$

where $STE$ denotes the space time exchange operator. As shown below, for problems where there is no preferred direction, but rather spatial symmetry, it may also be useful to define three exchange operations:

$$x^i \leftrightarrow x^j, \quad i = 1, 2, 3, \quad (1.5)$$

with associated operations $STE(i)$ analogous to $STE = STE(1)$ in Eqs. (1.2)–(1.4). The operations in Eqs. (1.2)–(1.5) may also be generalized to the case of an arbitrary four-vector with temporal and spatial components $A^0$ and $A^1$, respectively.

To clarify the meaning of the $STE$ operation, it is of interest to compare it with a different operator acting on space and time coordinates that may be called "space-time coordinate permutation" (STCP). Consider an equation of the form

$$f(x^0, x^1) = 0. \quad (1.6)$$

The $STE$ conjugate equation is

$$f(x^1, x^0) = 0. \quad (1.7)$$

This equation is different from (1.6) because $x^0$ and $x^1$ have different physical meanings. In the STCP operation, however, the values of the space and time coordinates are interchanged, but no new equation is generated. If $x^0 = a$ and $x^1 = b$ in Eq. (1.6), then the STCP operation applied to the latter yields

$$f(x^0 = b, x^1 = a) = 0. \quad (1.8)$$

This equation is identical in form to (1.6); only its parameters have different values.

The physical meaning of the universal parameter $V$, and its relation to the velocity of light, $c$, is discussed in the following section, after the derivation of the LT.

The plan of the paper is as follows. In the following section the LT is derived. In Sec. III, the LT is used to derive the KSRP. The space-time exchange properties of four-vectors and the related symmetries in Minkowski space are discussed in Sec. IV. In Sec. V the space-time exchange symmetries of Maxwell’s equations are used to derive electrodynamics (Ampère’s law) and magnetodynamics (the Faraday–Lenz law) from the Gauss laws of electrostatics and magnetostatics, respectively. A summary is given in Sec. VI.

II. DERIVATION OF THE LORENTZ TRANSFORMATION

Consider two inertial frames, $S$, $S'$, $S'$ moves along the common $x, x'$ axis of orthogonal Cartesian coordinate systems in $S, S'$ with velocity $v$ relative to $S$. The $y, y'$ axes are parallel. At time $t = t' = 0$ the origins of $S$ and $S'$ coincide. In general the transformation equation between the coordinate $x$ in $S$ of a fixed point on the $Ox$ axis and the coordinate $x'$ of the same point referred to the frame $S'$ is

$$x' = f(x, x^0, \beta), \quad (2.1)$$

where $\beta = v/V$ and $V$ is the universal constant introduced in Eq. (1.1). Differentiating Eq. (2.1) with respect to $x^0$, for fixed $x'$, gives

$$dx' \Bigg|_{x'} = 0 = dx \frac{\partial f}{\partial x^0} + \frac{\partial f}{\partial x^0} \frac{\partial x^0}{\partial x^0}' \quad (2.2)$$

Since

$$\frac{dx}{dx^0} \frac{1}{\sqrt{\gamma}} = \frac{v}{V} = \beta,$$

the function $f$ must satisfy the partial differential equation:

$$\beta \frac{\partial f}{\partial x} = - \frac{\partial f}{\partial x^0}. \quad (2.3)$$

A sufficient condition for $f$ to be a solution of Eq. (2.3) is that it is a function of $x - \beta x^0$. Assuming also $f$ is a differentiable function, it may be expanded in a Taylor series:

$$x' = \gamma(\beta)(x - \beta x^0) + \sum_{n=2} \alpha_n(\beta)(x - \beta x^0)^n. \quad (2.4)$$

Requiring either spatial homogeneity, $\gamma(\beta) = \gamma$, or that the LT is a unique, single-valued, function of its arguments, requires Eq. (2.4) to be linear, i.e.,

$$\alpha_2(\beta) = \alpha_3(\beta) = \cdots = 0$$

so that

$$x' = \gamma(\beta)(x - \beta x^0). \quad (2.5)$$

Spatial homogeneity implies that Eq. (2.5) is invariant when all spatial coordinates are scaled by any constant factor $K$. Noting that

$$\beta = \frac{1}{\sqrt{\gamma}} \left| \frac{dx}{dt} \right|_{x'} = \frac{1}{\sqrt{\gamma}} \left| \frac{d(-x)}{dt} \right|_{x'}, \quad (2.6)$$

and choosing $K = -1$ gives

$$-x' = \gamma(-\beta)(-x + \beta x^0). \quad (2.7)$$

Hence, Eq. (2.5) is invariant provided that

$$\gamma(-\beta) = \gamma(\beta), \quad (2.8)$$

i.e., $\gamma(\beta)$ is an even function of $\beta$.

Applying the space-time exchange operations $x \leftrightarrow x^0$, $x' \leftrightarrow (x^0)'$ to Eq. (2.5) gives

$$(x^0)' = \gamma(\beta)(x^0 - s x). \quad (2.9)$$
The transformation inverse to (2.9) may, in general, be written as:
\[ x^0 = \gamma(\beta')(x^0') - \beta'x'. \]  
(2.10)
The same inverse transformation may also be derived by eliminating \( x \) between Eqs. (2.5) and (2.9) and rearranging:
\[ x^0 = \frac{1}{\gamma(\beta)(1 - \beta^2)} (x^0' + \beta x'). \]  
(2.11)
Equations (2.10) and (2.11) are consistent provided that
\[ \gamma(\beta') = \frac{1}{\gamma(\beta)(1 - \beta^2)} \]  
(2.12)
and
\[ \beta' = -\beta. \]  
(2.13)
Equations (2.8), (2.12), and (2.13) then give
\[ \gamma(\beta) = \frac{1}{\sqrt{1 - \beta^2}}. \]  
(2.14)
Equations (2.5) and (2.9) with \( \gamma \) given by (2.14) are the LT equations for space-time points along the common \( x, x' \) axis of the frames \( S, S' \). They have been derived here solely from the symmetry condition (I) and the assumption of spatial homogeneity, without any reference to the Principle of Special Relativity.

The physical meaning of the universal parameter \( V \) becomes clear when the kinematical consequences of the LT for physical objects are worked out in detail. This is done, for example, in Ref. 6, where it is shown that the velocity of any massive physical object approaches \( V \) in any inertial frame in which its energy is much greater than its rest mass. The identification of \( V \) with the velocity of light, \( c \), then follows if it is assumed that light consists of massless (or almost massless) particles, the light quanta discovered by Einstein in his analysis of the photoelectric effect. That \( V \) is the limiting velocity for the applicability of the LT equations is, however, already evident from Eq. (2.14). If \( \gamma(\beta) \) is real, then \( \beta \leq 1 \), that is, \( u \leq V \).

**III. DERIVATION OF THE KINEMATICAL SPECIAL RELATIVITY PRINCIPLE**

The LT equations (2.5) and (2.9) and their inverses, written in terms of \( x, x' \); \( t, t' \), are
\[ x' = \gamma(x - vt), \]  
(3.1)
\[ t' = \gamma(t - \frac{vx}{V^2}), \]  
(3.2)
\[ x = \gamma(x' + vt'), \]  
(3.3)
\[ t = \gamma(t' + \frac{vx'}{V^2}). \]  
(3.4)
Consider now observers, at rest in the frames \( S, S' \), equipped with identical measuring rods and clocks. The observer in \( S' \) places a rod, of length \( l \), along the common \( x, x' \) axis. The coordinates in \( S' \) of the ends of the rod are \( x'_1, x'_2 \), where \( x'_2 - x'_1 = l \). If the observer in \( S \) measures, at time \( t \) in his own frame, the ends of the rod to be at \( x_1, x_2 \) then, according to Eq. (3.1):
\[ x'_1 = \gamma(x_1 - vt), \]  
(3.5)
\[ x'_2 = \gamma(x_2 - vt). \]  
(3.6)
Denoting by \( l_S \) the apparent length of the rod, as observed from \( S \) at time \( t \), Eqs. (3.5) and (3.6) give
\[ l_S = x_2 - x_1 = \frac{1}{\gamma} (x'_1 - x'_2) = \frac{l}{\gamma}. \]  
(3.7)
Suppose that the observer in \( S' \) now makes reciprocal measurements \( x'_1, x'_2 \) of the ends of a similar rod, at rest in \( S \), at time \( t' \). In \( S \) the ends of the rod are at the points \( x_1, x_2 \), where \( l = x_2 - x_1 \). Using Eq. (3.3)
\[ x_1 = \gamma(x'_1 + vt'), \]  
(3.8)
\[ x_2 = \gamma(x'_2 + vt') \]  
(3.9)
and, corresponding to (3.7), there is the relation
\[ l_{S'} = x'_2 - x'_1 = \frac{1}{\gamma} (x_2 - x_1) = \frac{l}{\gamma}. \]  
(3.10)
Hence, from Eqs. (3.7) and (3.10)
\[ l_S = l_{S'} = \frac{1}{\gamma}, \]  
(3.11)
so that reciprocal length measurements yield identical results.

Consider now a clock at rest in \( S' \) at \( x' = 0 \). This clock is synchronized with a similar clock in \( S \) at \( t = t' = 0 \), when the spatial coordinate systems in \( S \) and \( S' \) coincide. Suppose that the observer at rest in \( S \) notes the time \( t \) recorded by his own clock, when the moving clock records the time \( \tau \). At this time, the clock which is moving along the common \( x, x' \) axis with velocity \( v \) will be situated at \( x = vt \). With the definition \( \tau_S = \tau \), and using Eq. (3.2),
\[ \tau = \gamma \left( \frac{\tau_S - \frac{vx}{V^2}}{V^2} \right) = \gamma \tau_S \left( 1 - \frac{v^2}{V^2} \right) = \frac{\tau_S}{\gamma}. \]  
(3.12)
If the observer at rest in \( S' \) makes a reciprocal measurement of the clock at rest in \( S \), which is seen to be at \( x' = -vt' \) when it shows the time \( \tau \), then according to Eq. (3.4) with \( \tau_{S'} = \tau' \),
\[ \tau = \gamma \left( \tau_{S'} + \frac{vx'}{V^2} \right) = \gamma \tau_{S'} \left( 1 - \frac{v^2}{V^2} \right) = \frac{\tau_{S'}}{\gamma}. \]  
(3.13)
Equations (3.12) and (3.13) give
\[ \tau_S = \tau_{S'} = \gamma \tau. \]  
(3.14)
Equations (3.11) and (3.14) prove the Kinematical Special Relativity Principle as stated above. It is a necessary consequence of the LT.

**IV. GENERAL SPACE-TIME EXCHANGE SYMMETRY PROPERTIES OF FOUR-VECTORS. SYMMETRIES OF MINKOWSKI SPACE**

The LT was derived above for space-time points lying along the common \( x, x' \) axis, so that \( x = [x] \). However, this restriction is not necessary. In the case that \( \vec{x} = (x^1, x^2, x^3) \), then \( x \) and \( x' \) in Eq. (2.5) may be replaced by \( x = \vec{x} \cdot \vec{u} / |\vec{u}| \) and \( x' = \vec{x}' \cdot \vec{u} / |\vec{u}| \), respectively, where the 1-axis is chosen
parallel to \( \vec{\sigma} \). The proof proceeds as before with the space-time exchange operation defined as in Eqs. (1.2)–(1.4). The additional transformation equations,

\[
y' = y, \quad (4.1)
\]
\[
z' = z, \quad (4.2)
\]

follow from spatial isotropy.\(^1\)

In the above derivation of the LT, application of the STE operator generates the LT of time from that of space. It is the pair of equations that is invariant with respect to the STE operation. Alternatively, as shown below, by a suitable change of variables, equivalent equations may be defined that are manifestly invariant under the STE operation.

The four-vector velocity \( U \) and the energy-momentum four-vector \( P \) are defined in terms of the space-time four-vector,\(^2\)

\[
X = (\sqrt{V \tau}; x, y, z) = (x^0; x^1, x^2, x^3),
\]

by the equations

\[
U = \frac{dX}{d\tau},
\]

\[
P = mV,
\]

where \( m \) is the Newtonian mass of the physical object and \( \tau \) is its proper time, i.e., the time in a reference frame in which the object is at rest. Since \( \tau \) is a Lorentz invariant quantity, the four-vectors \( U, P \) have identical LT properties to \( X \). The properties of \( U, P \) under the STE operation follow directly from Eqs. (1.2) and (1.3) and the definitions (4.4) and (4.5).

Writing the energy-momentum four-vector as

\[
P = \left[ \frac{E}{V}; p_0, 0, 0 \right] = (p^0, p^1, 0, 0),
\]

the STE operations: \( p^0 \leftrightarrow p^1, (p^0) \leftrightarrow (p^1) \) generate the LT equation for energy

\[
(p^0)' = \gamma (p^0 - \beta p^1)
\]

from that of momentum

\[
(p^1)' = \gamma (p^1 - \beta p^0)
\]
or vice versa.

The scalar product of two arbitrary four-vectors \( C, D \),

\[
C \cdot D \equiv C^0 D^0 - \vec{C} \cdot \vec{D},
\]

can, by choosing the \( x \)-axis parallel to \( \vec{C} \) or \( \vec{D} \), always be written as:

\[
C \cdot D = C^0 D^0 - C^1 D^1.
\]

Defining the STE exchange operation for an arbitrary four-vector in a similar way to Eqs. (1.2) and (1.3), then the combined operations \( C^0 \rightarrow C^1, D^0 \rightarrow D^1 \) yield

\[
C \cdot D \rightarrow C^1 D^1 - C^0 D^0 = -C \cdot D.
\]

The four-vector product changes sign, and so the combined STE operation is equivalent to a change in the sign convention of the metric from space-like to time-like (or vice versa), hence the corollary (II) in Sec. I.

The LT equations take a particularly simple form if new variables are defined which have simple transformation properties under the STE operation. The variables are

\[
x_+ = \frac{x^0 + x^1}{\sqrt{2}}, \quad (4.12)
\]
\[
x_+ = \frac{x^0 - x^1}{\sqrt{2}}.
\]

\( x_+ \), \( x_- \) have, respectively, even and odd ‘‘STE parity’’:

\[
STE x_+ = x_+,
\]
\[
STE x_- = -x_-.
\]

The manifestly STE invariant LT equations expressed in terms of these variables are

\[
x_+ = \alpha x_+,
\]
\[
x_- = \frac{1}{\alpha} x_-,
\]

where

\[
\alpha = \sqrt{1 - \beta^2} \quad (1 + \beta^2).
\]

Introducing similar variables for an arbitrary four-vector,

\[
C_+ = \frac{C^0 + C^1}{\sqrt{2}},
\]
\[
C_- = \frac{C^0 - C^1}{\sqrt{2}},
\]

the 4-vector scalar product of \( C \) and \( D \) may be written as

\[
C \cdot D = C_+ D_+ + C_- D_-.
\]

In view of the LT equations (4.16) and (4.17), \( C \cdot D \) is manifestly Lorentz invariant. The transformations (4.12), (4.13) and (4.19), (4.20) correspond to an anti-clockwise rotation by \( 45^\circ \) of the axes of the usual \( ct \) versus \( x \) plot. The \( x_+, x_- \) axes lie along the light cones of the \( x-ct \) plot (see Fig. 1).

The LT equations (4.16) and (4.17) give a parametric representation of a hyperbola in \( x_+, x_- \) space. A point on the latter corresponds to a particular space-time point as viewed in a frame \( S \). The point \( x_+ = x_- = 0 \) corresponds to the space-time origin of the frame \( S' \) moving with velocity \( \beta c \) relative to \( S \). A point at the spatial origin of \( S' \) at time \( t' = \tau \) will be seen by an observer in \( S \) as \( \beta \) (and hence \( \alpha \)) varies, to lie on one of the hyperbolae \( H_{++}, H_{--} \) in Fig. 1:

\[
x_+ x_- = \frac{c^2 \tau^2}{2},
\]

with \( x_+, x_- > 0 \) if \( \tau > 0(H_{++}) \) or \( x_+, x_- < 0 \) if \( \tau < 0(H_{--}) \). A point along the \( x'_+ \)-axis at a distance \( s \) from the origin, at \( t' = 0 \), lies on the hyperbolae \( H_{++}, H_{--} \) :

\[
x_+ x_- = -\frac{s^2}{2},
\]

with \( x_+ > 0, x_- < 0 \) if \( s > 0(H_{++}) \) or \( x_+ < 0, x_- > 0 \) if \( s < 0(H_{++}) \). As indicated in Fig. 1 the hyperbolae (4.22) correspond to the past (\( \tau < 0 \)) or the future (\( \tau > 0 \)) of a space-time point at the origin of \( S \) or \( S' \), whereas (4.23) corresponds to the ‘‘elsewhere’’ of the same space-time points, that is, the manifold of all space-time points that are causally
In the following, Maxwell’s equations are written in Heaviside–Lorentz units with $V=c=1$. The four-vector potential $A = (A^0; \vec{A})$ is related to the electromagnetic field tensor $F^{\mu\nu}$ by the equation

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$  \hspace{1cm} (5.1)

where

$$\partial^\mu = \left( \frac{\partial}{\partial t}; - \vec{\nabla} \right) = (\partial_0; - \vec{\nabla}).$$  \hspace{1cm} (5.2)

The electric and magnetic field components, $E^k$ and $B^k$, respectively, are given, in terms of $F^{\mu\nu}$, by the equations

$$E^k = F^{k0},$$  \hspace{1cm} (5.3)

$$B^k = - \epsilon_{ijk} F^{ij}.$$  \hspace{1cm} (5.4)

A time-like metric is used with $C_0 = C^0 = C_1 = C^1 = -C_1$, etc., with summation over repeated contravariant (upper) and covariant (lower) indices understood. Repeated Greek indices are summed from 1 to 4 and Roman ones from 1 to 3.

The transformation properties of contravariant and covariant four-vectors under the STE operation are now discussed. They are derived from the general condition that four-vector products change sign under the STE operation [Eq. (4.11)]. The four-vector product $A^0 B^1 = C^0 D^1$, written in terms of contravariant and covariant four-vectors, as

$$C \cdot D = C^0 D_0 + C^1 D_1.$$  \hspace{1cm} (5.5)

Assuming that the contravariant four-vector $C^\mu$ transforms according to Eqs. (1.2) and (1.3), i.e.,

$$C^0 \rightarrow C^1,$$  \hspace{1cm} (5.6)

the covariant four-vector $D_\mu$ must transform as:

$$D_0 \rightarrow - D_1.$$  \hspace{1cm} (5.7)

in order to respect the transformation property

$$C \cdot D \rightarrow C \cdot D.$$  \hspace{1cm} (5.8)

of four-vector products under STE.

It remains to discuss the STE transformation properties of $\partial^\mu$ and the four-vector potential $A^\mu$. In view of the property of $\partial^\mu$ : $\partial^1 = - \partial_1 = - \partial/\partial x$ [Eq. (5.2)], which is similar to the relation $C_1 = - C_0$ for a covariant four-vector, it is natural to choose for $\partial^\mu$ a STE transformation similar to Eq. (5.7):

$$\partial^0 \rightarrow - \partial^1,$$  \hspace{1cm} (5.9)

and hence, in order that $\partial^\mu \partial_\mu$ change sign under STE:

$$\partial_0 \rightarrow \partial_1.$$  \hspace{1cm} (5.10)

This is because it is clear that the appearance of a minus sign in the STE transformation equation (5.7) is correlated with the minus sign in front of the spatial components of a covariant four-vector, not whether the Lorentz index is an upper or lower one. Thus $\partial^\mu$ and $\partial_\mu$ transform in an “anomalous” manner under STE as compared to the convention of Eqs. (5.6) and (5.7). In order that the four-vector product $\partial_\mu A^\mu$ respect the condition (5.8), $A^\mu$ and $A_\mu$ must then transform under STE as:

$$A^0 \rightarrow A^1$$  \hspace{1cm} (5.11)

and
As \( A_0 \rightarrow A_1 \),

\[
A_0 = A_1,
\]

respectively. That is, they transform in the same way as \( \partial^\mu \) and \( \partial_\mu \), respectively.

Introducing the four-vector electromagnetic current \( j^\mu = (\rho, j^j) \), Gauss’ law of electrostatics may be written as:

\[
\nabla \cdot \vec{E} = \rho = j^0,
\]

(5.13)
or, in the manifestly covariant form,

\[
(\partial_\mu j^\mu) A^0 - \partial^0 (\partial_\mu A^\mu) = j^0.
\]

(5.14)

This equation is obtained by writing Eq. (5.13) in covariant notation using Eqs. (5.1) and (5.3) and adding to the left side the identity:

\[
\partial_\mu (\partial^\mu A^0 - \partial^0 A^\mu) = 0.
\]

(5.15)

Applying the space-time exchange operation to Eq. (5.14), with index exchange \( 0 \rightarrow 1 \) [noting that \( \partial^0 A^0 \) transform according to Eqs. (5.9) and (5.11), \( j^0 \) according to (5.6), and that the scalar products \( \partial_\mu \partial^\mu \) and \( \partial_\mu A^\mu \) change sign] yields the equation

\[
(\partial_\mu j^\mu) A^0 - \partial^0 (\partial_\mu A^\mu) = j^1.
\]

(5.16)

The spatial part of the four-vector products on the left side of Eq. (5.16) is

\[
\partial_1 (\partial^0 A^1 - \partial^1 A^0) = \partial_1 F_1 = \partial_3 B^3 - \partial_3 B^2 = (\nabla \times \vec{B})^1,
\]

(5.17)

where Eqs. (5.1) and (5.4) have been used. The time part of the four-vector products in Eq. (5.16) yields, with Eqs. (5.1) and (5.3),

\[
\partial_0 (\partial^0 A^1 - \partial^1 A^0) = -\frac{\partial E^1}{\partial t}.
\]

(5.18)

Combining Eqs. (5.16)–(5.18) gives

\[
(\nabla \times \vec{B})^1 - \frac{\partial E^1}{\partial t} = j^1.
\]

(5.19)

Combining Eq. (5.19) with the two similar equations derived by the index exchanges \( 0 \rightarrow 2, 0 \rightarrow 3 \) in Eq. (5.14) gives

\[
(\nabla \times \vec{B}) - \frac{\partial \vec{E}}{\partial t} = \vec{j}.
\]

(5.20)

This is Ampère’s law, together with Maxwell’s displacement current.

The Faraday-Lenz law is now derived by applying the space-time exchange operation to the Gauss law of magnetostatics:

\[
\nabla \cdot \vec{B} = 0.
\]

(5.21)

Introducing Eqs. (5.4) and (5.1) into Eq. (5.21) gives

\[
\partial_1 (\partial^3 A^2 - \partial^2 A^3) + \partial_2 (\partial^3 A^1 - \partial^3 A^1) + \partial_3 (\partial^3 A^1 - \partial^3 A^2) = 0.
\]

(5.22)

Making the exchange \( 1 \rightarrow 0 \) of space-time indices in Eq. (5.22) and noting that \( \partial_1 \) transforms according to Eq. (5.10), whereas \( \partial^0 A^j \) transform as in Eqs. (5.9) and (5.11), respectively, gives

\[
\partial_0 (\partial^3 A^2 - \partial^2 A^3) + \partial_2 (\partial^3 A^0 - \partial^3 A^0) + \partial_3 (\partial^3 A^0 - \partial^3 A^0) = 0.
\]

(5.23)

Using Eqs. (5.1)–(5.4), Eq. (5.23) may be written as:

\[
\frac{\partial B^1}{\partial t} + \partial_2 E^3 - \partial_3 E^2 = 0,
\]

(5.24)

or, in three-vector notation,

\[
(\nabla \times \vec{E})^1 = -\frac{\partial B^1}{\partial t}.
\]

(5.25)

The space-time exchanges \( 2 \rightarrow 0, 3 \rightarrow 0 \) in Eq. (5.22) yield, in a similar manner, the two- and three-components of the Faraday–Lenz law:

\[
(\nabla \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t}.
\]

(5.26)

VI. SUMMARY AND DISCUSSION

In this paper the Lorentz transformation for points lying along the common \( x, x' \) axis of two inertial frames has been derived from only two postulates: \( (i) \) the symmetry principle (I), and \( (ii) \) the homogeneity of space. This is the same number of axioms as used in Ref. 6 where the postulates were the Kinematical Special Relativity Postulate and the uniqueness condition. Since both spatial homogeneity and uniqueness require the LT equations to be linear, the KSRP of Ref. 6 has here, essentially, been replaced by the space-time symmetry condition (I).
Although postulate (I) and the KRSP play equivalent roles in the derivation of the LT, they state in very different ways the physical foundation of special relativity. Postulate (I) is a mathematical statement about the structure of the equations of physics, whereas the KSRP makes, instead, a statement about the relation between space-time measurements performed in two different inertial frames. It is important to note that in neither case do the dynamical laws describing any particular physical phenomenon enter into the derivation of the LT.

Choosing postulate (I) as the fundamental principle of special relativity instead of the Galilean Relativity Principle, as in the traditional approach, has the advantage that a clear distinction is made, from the outset, between classical and relativistic mechanics. Both the former and the latter respect the Galilean Relativity Principle but with different laws. On the other hand, only relativistic equations, such as the LT or Maxwell’s equations, respect the symmetry condition (I).

The teaching of, and hence the understanding of, special relativity differs greatly depending on how the parameter $V$ is introduced. In axiomatic derivations of the LT, which do not use Einstein’s second postulate, a universal parameter $V$ with the dimensions of velocity necessarily appears at an intermediate stage of the derivation.\textsuperscript{19} Its physical meaning, as the absolute upper limit of the observed velocity of any physical object, only becomes clear on working out the kinematical consequences of the LT.\textsuperscript{6} If Einstein’s second postulate is used to introduce the parameter $c$, as is done in the vast majority of textbook treatments of special relativity, justified by the empirical observation of the constancy of the velocity of light, the actual universality of the theory is not evident. The misleading impression may be given that special relativity is an aspect of classical electrodynamics, the domain of physics in which it was discovered.

Formulating special relativity according to the symmetry principle (I) makes clear the space-time geometrical basis\textsuperscript{5} of the theory. The universal velocity parameter $V$ must be introduced at the outset in order even to define the space-time exchange operation. Unlike the Galilean Relativity Principle, the symmetry condition (I) gives a clear test of whether any physical equation is a candidate to describe a universal law of physics. Such an equation must either be invariant under space-time exchange or related by the exchange operation to another equation that also represents a universal law. The invariant amplitudes of quantum field theory are an example of the former case, while the LT equations for space and time correspond to the latter. Maxwell’s equations are examples of dynamical laws that satisfy the symmetry condition (I). The laws of electrostatics and magnetostatics (Gauss’ law for electric and magnetic charges) are related by the space-time exchange symmetry to the laws of electrodynamics (Amperé’s law) and magnetodynamics (the Faraday–Lenz law), respectively. The four-vector character\textsuperscript{20} of the electromagnetic potential is essential for these symmetry relations.\textsuperscript{21}

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\bibitem{3} W. v Ignatowsky, Arch. Math. Phys. Lpz. 17, 1 (1910); 18, 17 (1911); Phys. Z. 11, 972 (1910); 12, 779 (1911).
\bibitem{11} The positive sign for $\gamma$ is taken in solving Eq. (2.12). Evidently $\gamma \rightarrow 1$ as $\beta \rightarrow 0$.
\bibitem{14} See, for example, J. L. Aitchison and A. J. G. Hey Gauge Theories in Particle Physics (Hilger, London, 1982), Appendix C.
\bibitem{15} See, for example, S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972), p. 36.
\bibitem{16} See, for example, Eq. (2.36) of Ref. 6.
\bibitem{17} For a recent discussion of the physical meaning of the three-vector magnetic potential see M. D. Semon and J. R. Taylor, “Thoughts on the magnetic vector potential,” Am. J. Phys. 64, 1361–1369 (1996).
\bibitem{18} It is often stated in the literature that the potentials $\phi, A$ are introduced only for “reasons of mathematical simplicity” and “have no physical meaning.” See, for example: F. Röhrlich, Classical Charged Particles (Addison-Wesley, Reading, MA, 1990), pp. 65–66. Actually, the underlying space-time symmetries of Maxwell’s equations can only be expressed by using the four-vector character of $A$. Also the minimal electromagnetic interaction in the covariant formulation of relativistic quantum mechanics, which is the dynamical basis of quantum electrodynamics, requires the introduction of a quantum field for the photon that has the same four-vector nature as the electromagnetic potential.
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