Statistical Mechanics (Solve only 2 questions out of 3)

1) Consider a gas of noninteracting (spinless) bosons with energy $\epsilon = c |\vec{p}|$ contained in a box of volume $V$ in 3-dimensions.
(a) Calculate the grand potential $G = -k_B T \ln Q$, and the density $n = N/V$, at a chemical potential $\mu$. Express your answers in terms of $g_m(z)$ where $z = e^{\beta \mu}$ ($\beta = 1/k_B T$) and

$$g_m = \frac{1}{(m-1)!} \int_0^\infty dx \frac{x^{m-1}}{z^m e^x - 1}$$

(b) Calculate the gas pressure $P$, its energy $E$, and compare the ratio $E/(PV)$ to the classical value.
(c) Find the critical temperature $T_c(n)$ for Bose-Einstein condensation.

2) Suppose that ions with spin $s = 1/2$, embedded in certain weird media, undergo a transformation from a paramagnetic to a ferromagnetic state. A phenomenological temperature dependence for the molar heat capacity associated with these spins is given by

$$C(T) = C_0 \left[ \left( \frac{T}{T_2} \right)^n - \left( \frac{T_1}{T_2} \right)^n \right]$$

The transition takes place between $T_1$ and $T_2$. A fully ferromagnetic state exists below $T_1$. Above $T_2$, a fully disordered paramagnetic state exists. The nature of the weird medium determines the parameters $n, T_1$, and $T_2$. However, $C_0$ is fully determined by these parameters and fundamental constants.

a) What is the entropy per mole of the spins below $T_1$?

b) What is the entropy per mole of the spins above $T_2$?

c) Using the phenomenological molar heat capacity given above, calculate the entropy per mole between $T_1$ and $T_2$.

d) Find an expression for $C_0$ in terms of $n, T_1$, and $T_2$ using the above results.
3) You are given a system of two identical particles which may occupy any of the three energy levels

\[ \varepsilon_n = n\varepsilon, \quad n = 0, 1, 2 \]

The lowest energy state, \( \varepsilon_0 = 0 \), is doubly degenerate. The system is in thermal equilibrium at temperature \( T \). For each of the following cases determine the partition function and the energy and carefully enumerate the configurations:

a) The particles obey Fermi statistics
b) The particles obey Bose statistics
c) The particles obey Boltzmann statistics. (Now distinguishable)

Mathematical Physics (Solve only 2 questions out of 3)

1) a) For a three-dimensional potential \( V(\vec{r}) \), let its Fourier transform be denoted by \( \hat{V}(\vec{k}) \), write down the Fourier and inverse Fourier transform relations between these two functions.

b) If \( V(\vec{r}) = \frac{e^{-ar}}{r} \) with \( a > 0 \), what is the name given to this potential? Find its Fourier transform \( \hat{V}(\vec{k}) \).

c) Determine the Fourier transform of the Coulomb potential.

2) Consider four Hermitian 2x2 matrices \( I, \sigma_1, \sigma_2, \sigma_3 \), where \( I \) is the unit matrix, and the others anticommute, i.e. satisfy

\[ \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \]

Prove the following without using a specific representation of form of the matrices:

a) Prove that \( Tr(\sigma_i) = 0 \).

b) Show that the eigenvalues of \( \sigma_i \) are \( \pm 1 \) and \( \text{det}(\sigma_i) = -1 \).

c) Show that the four matrices are linearly independent and therefore that any 2x2 matrix can be expanded in terms of them.

d) From (c) we know that

\[ M = m_0I + \sum_{i=1}^{3} m_i \sigma_i \]

where \( M \) is any 2x2 matrix. Derive an expression for \( m_i (i = 0, 1, 2, 3) \).

3) Evaluate the following integral by the method of residues

\[ \int_{0}^{\pi/2} \frac{dx}{a + \sin^2 x} \quad |a| > 1 \]

Justify all the steps carefully.
Modern Physics (Solve only 1 question out of 2)

1) Calculate the density of states in various dimensions, i.e., for a free particle confined to boxes in one, two, and three dimensions. Also calculate the density of states for slab geometry, i.e., the box of a x L x L, where \( a \ll L \), and discuss this in terms of previous results.

2) Assume that a particle of mass \( m \) has to move on a cone, sides of which make an angle \( \theta \) with the horizontal (see figure below). Acceleration due to gravity is \( g \).
Using Bohr’s quantization rule \( L = n\hbar \), find the smallest possible kinetic energy the particle can have, and all allowed values of its height from the tip of the cone \( h_n \). 

![Diagram of particle on a cone](attachment:cone_diagram.png)
Quantum Mechanics (Solve only 3 questions out of 4)

1) Consider a particle that is scattered from a step potential plus a delta function (see the figure)

\[ V(x) = V_0 \Theta(x) + U_0 \delta(x), \]

where step function is defined as

\[ \Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases} \]

a) What are the units of \( V_0 \) and \( U_0 \)?

b) Find the boundary condition the wavefunction must satisfy at 0.

c) Assume that a particle is coming from \(-\infty\) with energy \( E > V_0 \), calculate the transmission coefficient \( T \).

2) Assume that a particle of mass \( m \) is in the ground state of an infinite potential well extending from 0 to \( a \). Suddenly at time \( t = 0 \), the well starts moving right with velocity \( v \). Calculate the probability that the particle will remain in the ground state of the moving well.

3) Consider three distinguishable particles A, B and C. All of them have spin 1. They form a composite particle, with total angular momentum \( J \).

\[ \vec{J} = \vec{S}_A + \vec{S}_B + \vec{S}_C \]

The state of the composite particle can be expressed in terms of the kets

\[ |S_A, m_A > \otimes |S_B, m_B > \otimes |S_C, m_C > \]

a) What are the possible values of \( J \)?

b) If the composite particle is in state \( J = 3; m_J = 2 \), what is the state of the system? What is the probability that particle A has its z component of spin \( m_A = 1 \)?

c) It is known that the composite particle formed from them is in a state of \( J=0 \). How many different (independent) wave functions can you write for this state, write all of them.
4) A particle of mass \( m \) moves in a three-dimensional harmonic oscillator well. The Hamiltonian is

\[
H = \frac{\vec{p}^2}{2m} + \frac{1}{2} m \omega^2 r^2
\]

a) Find the energy of the ground state and the first three excited states.

b) If eight identical non-interacting (spin-1/2) particles are placed in such a harmonic potential, find the ground state energy for the eight-particle system.

c) Write down the wavefunctions of the system.
Analytical Mechanics (Solve only 2 questions out of 3)

1) The classical interaction between two inert gas atoms, each of mass $m$, is given by the potential

$$V(r) = \frac{2A}{r^6} + \frac{B}{r^{1/2}}, \quad A, B > 0$$

a) Give the Hamiltonian for the system of the two atoms.
b) Describe completely the lowest energy classical state of the system.

2) A thin, uniform rod of length $2L$ and mass $M$ is suspended from a massless string of length $l$ tied to a nail. As shown in figure, a horizontal force $F$ is applied to the rod’s free end.

a) write the Lagrange equations for this system.
b) Draw a diagram to illustrate the initial motion of the rod.

3) A uniform solid cylinder of radius $R$ and mass $M$ rests on a horizontal plane and an identical cylinder rests on it, touching it on top. The upper cylinder is given an infinitesimal displacement so that both cylinders roll without slipping.

a) What is the Lagrangian of the system?
b) What are the constants of motion?
Freshman Physics (Solve only 1 question out of 2)

1) A triangle consists of steel rod of mass $M$, bent into the shape of an equilateral triangle of length $L$ on each side. This triangle is suspended from one of its corners.
   a) Show that the moment of inertia is $I = (ML^2)/2$ about an axis through this corner and perpendicular to its plane.
   b) A small disturbance is applied to the triangle and it starts an oscillatory motion about the same axis under gravitation. Determine the natural frequency of this oscillation for small amplitude oscillations.

2) A mass $M$ slides without friction on the roller coaster track shown below. The curved sections of the track have radius of curvature $R$. The mass begins its descent from a height $h$. At some value of $h$, the mass will begin to lose contact with the track at some point. Specify where on the track the mass loses the contact with the track and calculate the minimum value of $h$ for which this happens.
Electromagnetic Theory (Solve only 2 questions out of 3)

1) a) Derive the Poynting’s theorem (conservation of energy) in differential form starting from Maxwell’s equations assuming a medium having permittivity $\varepsilon$ and permeability $\mu$ with a current density $\vec{J}$. In the final expression identity and interpret all terms.
(Poynting’s vector is $\vec{S} = \vec{E} \times \vec{H}$)
b) Switching to source-free medium having the electric field vector given as $\vec{E} = \hat{x} E_0 \cos(kz - \omega t)$, verify the Poynting’s theorem.
Hint: $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot \nabla \times \vec{F} - \vec{F} \cdot \nabla \times \vec{G}$

2) As shown in the figure, you are given the not-so-parallel plate capacitor.
a) Neglecting the edge effects, when a voltage difference $V$ is placed across the two conductors, find the potential everywhere between the plates.
b) When this wedge is filled with a medium of dielectric constant $\varepsilon$, calculate the capacitance of the system in terms of the constants given.

3) A very long conducting pipe has a square cross section of its inside surface, with size $D$ as shown below. For from either end of the pipe is suspended a point charge located at the center of the square cross section.
a) Determine the electric potential at all points inside the pipe.
b) Give the asymptotic expression for this potential for points far from the point charge.
c) Sketch some electric field lines in a region for from the point charge.
1) Resolution is an important consideration in many areas of experimental physics. Observation of a physical process often requires that it be measured with sufficient accuracy and precision and the instrumentation has the required resolving power to separate closely spaced observables both in the space of the variable of interest (time, space, wavelength, temperature, etc.).

a) Explain what we mean by accuracy during a measurement during the detection of the Balmer series of hydrogen.

b) In the same context, what is precision?

c) Is it possible to have high precision but low accuracy?

d) Is it possible to have high accuracy but low precision?

e) Assume that δ and α lines of Balmer series are closely spaced in wavelength examples of which are shown below. Which spectrum is at the limit of resolution and why? Explain the criteria you use to make the decision.

2) A radioactive source is measured N=10 times with time interval Δt=1 min. The data are given below.

a) What kind of probability distribution would you expect these data to obey?

b) What is the average and spread of data points.

c) What is the uncertainty in the mean?

d) If we make an additional measurement for a Δt=10 min interval, calculate the mean and the uncertainty in the mean for all the experiments.
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