Statistical Mechanics (Solve only 2 questions out of 3)

1) Rotational motion of a diatomic molecule is specified by two angular variables $\theta, \phi$ and the corresponding canonical conjugate momenta $p_\theta, p_\phi$.

Assuming the form of the kinetic energy of the rotational motion to be

$$E_{rot} = \frac{1}{2I} p_\theta^2 + \frac{1}{2I \sin^2 \theta} p_\phi^2$$

calculate the rotational partition function, $Z_{rot}(T)$.

Using this partition function, $Z_{rot}(T)$, calculate the entropy and specific heat.

2) For a D-dimensional medium in which the lowest-lying vibrational modes of wavelength $\lambda$ have frequencies given by a dispersion relation of the form

$$\omega \sim \lambda^{-s}$$

Find the expressions for the internal energy and the specific heat at constant volume using Debye approximation. What is the behavior of $C_v(T)$ near $T=0$?

3) One dimensional Ising Model: N spin $\frac{1}{2}$ particles are lined up on a straight line. Only nearest neighbors interact. Spin of each particle can be either up, or down. When the spins of nearest neighbors are both up, or both down, their interaction energy is $J$. When one is up and one is down, the interaction energy is $-J$.

Find the partition function for this system at temperature $T$.

Using this partition function, calculate the entropy and specific heat.

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For your information:

$$\int_{-\infty}^{\infty} dke^{-\alpha k^2} = \sqrt{\frac{\pi}{\alpha}}$$

The binomial formula: $(1 + x)^{N-1} = \sum_{n=0}^{N-1} \frac{(N-1)!}{n!(N-1-n)!} x^n$
Mathematical Physics (Solve only 2 questions out of 3)

1) A classical particle (mass m) is in a one-dimensional symmetrical potential well described by

\[ V(x) = Ax^{2n} \]

a) Investigate the behavior of the potential well as \( n \to \infty \). Obtain the period of the classical motion from the inspection of this limiting situation.

b) The period of the motion is given by

\[ T = \sqrt{\frac{2\pi m}{En^2 \left( \frac{E}{A} \right)^{\frac{1}{n}}} \frac{\Gamma \left( \frac{1}{2n} \right)}{\Gamma \left( \frac{1}{2n} + \frac{1}{2} \right)}} \]

Determine the limit of the above expression as \( n \to \infty \), and compare the result with that of part (a).

2) Generating function \( F \) for the Legendre Polynomials is given as:

\[ F(x, h) = \frac{1}{\sqrt{1 - 2hx + h^2}} = \sum_{n=0}^{\infty} h^n P_n(x) \]

Using this generating function or by other means

a) Calculate the first 3 Legendre polynomials \( P_0(x), P_1(x), P_2(x) \).

b) Calculate the definite integral

\[ \int_0^{\pi} d\theta \frac{\sin(\theta)(1 + A\cos(\theta))}{\sqrt{1 + 2A\cos(\theta) + A^2}} \quad (A > 1) \]

c) Show that the series

\[ S = \sum_{n=0}^{\infty} \frac{P_n(0)}{n+1} \]

can be conformed to an integral representation given by

\[ S = \int_0^{\infty} \frac{dx}{\sqrt{1 + e^{2x}}} \]

3) Evaluate the following integral

\[ \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 4x + 5} \, dx \]

Justify all the steps carefully.
1) A mass $m$ can move without friction along a circular wire (see figure). The wire rotates around the vertical diameter (the $z$ axis) with a constant angular velocity $\omega$. The mass $m$ is affected by the gravitational force downwards in the figure. Let $\theta$ be the angle between the vertical direction and the mass $m$ according to the figure.

a) Derive the equation of motion for $\theta$.
b) For low angular velocities, $\theta = 0$ is a stable equilibrium point, whereas it is unstable for high angular velocities. Determine the critical angular velocity $\omega_c$ that separates these two cases.
c) When $\omega < \omega_c$, only $\theta = 0$ and $\theta = \pi$ are equilibrium points, whereas when $\omega > \omega_c$ there is one more equilibrium point. Determine this point!

2) a) If $f, g$ and $h$ are functions of the canonical variables, show the following properties for the Poisson brackets,

\[
\{f, gh\} = g\{f, h\} + \{f, g\}h \\
\{fg, h\} = f\{g, h\} + \{f, h\}g
\]

b) Consider a particle in three dimensions that move in the potential

\[
U = \alpha z^2 e^{\beta x^2 + \gamma y^2} \quad ; \quad \alpha, \beta, \gamma = \text{const.} \quad \alpha \neq 0
\]

Determine a condition on $\beta$ and $\gamma$ such that the $z$ component of the angular momentum is conserved (for arbitrary initial conditions).

c) Calculate the moment of inertia $I$ of a homogeneous disk (mass $M$, radius $R$, thickness $d$) for a rotation about an axis normal (orthogonal) to the disk and through its center.
3) Consider a planar mathematical pendulum with the length $l$ and mass $m$. The thread goes through a hole and is pulled through this hole by an external force with a constant speed $\alpha$. The length of the pendulum thus decreases with time and can be written $l(t) = l_0 - \alpha t$.

Determine the Hamiltonian and the energy for the system. Is the Hamiltonian a constant of motion? Is the energy a constant of motion? Discuss your results.
1) Calculate the capacitance $C$ of a spherical capacitor of inner radius $a$ and outer radius $b$, which is filled with a dielectric varying as:

$$\varepsilon = \varepsilon_0 + \varepsilon_1 \cos^2(\theta)$$

where $\theta$ is the azimuthal polar angle in spherical polar coordinates.

2) A humpty-dumpty is formed from a semisphere of radius $r$ and cone of height $h$ from homogeneous material. What is maximum value of the height of the cone for stable equilibrium?
Electromagnetic Theory (Solve only 2 questions out of 3)

1) A monochromatic electromagnetic uniform plane wave propagates in unbounded free space along the +y direction. Its wavelength is given as 1 m and its time-averaged power flux is 1 W/m². The wave is right circularly polarized (also known as negative helicity).
   a) Find the electric and magnetic field vectors as a function of time and position.
   b) What is the time-averaged energy density of this EM wave?

2) A circular loop of wire of radius, $a$, is placed on the x-y plane, with its center at the origin. A current $I(t) = I_0 \cos \omega t$ flows within the wire.
   Starting with the expression for the vector potential in the far radiation zone, determine the vector potential and the radiating fields.
   What do you expect the angular radiation distribution to look like?

3) a) Starting from Maxwell equations for $E$, $D$, $B$ and $H$,
   \[
   \nabla \times \vec{H} = \frac{1}{c} \frac{\partial}{\partial t} \vec{D} + \frac{4\pi}{c} \vec{j} \\
   \nabla \cdot \vec{B} = 0 \\
   -\nabla \times \vec{E} = \frac{1}{c} \frac{\partial}{\partial t} \vec{B} \\
   \nabla \cdot \vec{D} = 4\pi \rho
   \]
   derive relations between these quantities at an interface of two different dielectric materials.
   b) Consider an interface of two dielectrics with permeabilities $\varepsilon_1$ and $\varepsilon_2$. Given the angle $\theta_1$ between the field $E_1$ and the normal to the interface, find the angle $\theta_2$ in the other dielectric.
   c) Consider an interface of two materials with magnetic permeabilities $\mu_1$ and $\mu_2$. Given the angle $\theta_1$ between the field $H_1$ and the normal to the interface, find the angle $\theta_2$ in the other material.

For your information:

\[
\int_0^{2\pi} e^{i\alpha \cos x} \cos(nx) dx = i^n \pi J_n(\alpha)
\]
Experimental Physics (Solve only 1 question out of 2)

1) Describe mechanisms of temperature measurement using
   a) a constant volume monometer
   b) a thermocouple
   c) an optical pyrometer

2) a) What are the sources of electronic noise? Explain
    b) Explain radioactive decay. Derive an expression for count rate, define half-life and mean-life.
1) A beam of atoms emerges from an oven that is at a temperature $T$. The distribution of the speeds of the atoms in the beam is proportional to $v^3 \exp(-mv^2/2kT)$.
   (a) Find the distribution of de Broglie wavelengths of the atoms,
   (b) the most probable de Broglie wavelength.

2) Estimate, using the uncertainty relation, the energy of the ground state of a two-electron atom of nuclear charge $Z$. 

Modern Physics (Solve only 1 question out of 2)
Quantum Mechanics (Solve only 3 questions out of 4)

1) A spin-1/2 particle of mass m moves in spherical harmonic oscillator potential 
\[ V = \frac{1}{2} m \omega^2 r^2 \] and subject to an interaction \( \lambda \hat{\sigma} \hat{r} \) (spin orbit forces are to be ignored).

The net Hamiltonian is therefore 
\[ H = H_0 + H_1 \]
where
\[ H_0 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2, \quad H_1 = \lambda \hat{\sigma} \hat{r} \]

Calculate the shift of the ground state energy through second order in the perturbation \( H_1 \).

2) Consider the one dimensional Schrödinger equation with a delta function potential:
\[ \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + U_0 \delta(x) \psi(x) = E \psi(x) \]

a) What are the units of \( U_0 \)?
b) What are the two boundary conditions that the wavefunction has to satisfy at \( x=0 \)?
c) Calculate the probability that a particle with energy \( E \) coming from the left will pass through the barrier. (This is known as the transmission coefficient.)

3) A particle of spin one is subject to the Hamiltonian 
\[ H = AS_z + BS_x^2 \]

where A and B are constants. Calculate the energy levels of this system. If at time zero \( t=0 \) the spin is in a eigenstate of \( S \) with \( S_z = +\hbar \), calculate the expectation value of the spin at time \( t \).

4) Consider a three-level system describe by the Hermitian Hamiltonian 
\[ H = H_0 + \lambda H_1 \]

where \( \lambda \) is a real number. The eigenstates of \( H_0 \) are \( |1\rangle, |2\rangle, \text{ and } |3\rangle \), and
\[ H_0 |1\rangle = 0 \]
\[ H_0 |2\rangle = \Delta |2\rangle \]
\[ H_0 |3\rangle = \Delta |3\rangle \]

a) Write down the most general 3x3 matrix representation of \( H_1 \) in the \{ |1\rangle, |2\rangle, |3\rangle \} basis.
b) When the spectrum of \( H \) is computed using perturbation theory, it is found that the eigenstates of \( H \) to lowest order in \( \lambda \) are \( \{ |1\rangle, |\pm\rangle = \frac{1}{\sqrt{2}} (|2\rangle \pm |3\rangle) \} \).
And the corresponding eigenvalues are

\[ E_1 = -\frac{\lambda^2}{\Delta} + O(\lambda^3) \]

\[ E_+ = \Delta + \lambda + \frac{\lambda^2}{\Delta} + O(\lambda^3) \]

\[ E_- = \Delta - \lambda + O(\lambda^3) \]

where \( O(\lambda^3) \) means order of \( (\lambda^3) \), i.e. third power of \( \lambda \).

Determine as many of the matrix elements of \( H_1 \) from part (a) as you can.

**For your information:**

i) The Pauli matrices are

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

ii) For 1D Harmonic oscillator lowering and raising operators are:

\[ A = \sqrt{\frac{m\omega}{2\hbar}} x + i \frac{p}{\sqrt{2m\omega\hbar}} \]

\[ A^\dagger = \sqrt{\frac{m\omega}{2\hbar}} x - i \frac{p}{\sqrt{2m\omega\hbar}} \]