Statistical Mechanics

1) Specific heat due to nuclear spin

A solid contains N mutually non-interacting nuclei of spin S=1. Each nucleus can therefore be in any of the three quantum states m=+1,0,-1. If the nucleus is in state m=+1 or m=-1, it has the same energy \( E=\Delta >0 \). If it is in state m=0, its energy is \( E=0 \).

a) Calculate the partition function for this system.
b) Derive an expression for the entropy at temperature \( T \).
c) Find the specific heat for this system at high temperatures \( \Delta \ll kT \).

2) A basic set of \( r \) points in space forms an elementary cell. If the elementary cell is repeated \( N \) times, we get a lattice. Assume that the atoms are located in the points \( rN \) and each atom is a three dimensional oscillator. Each of \( 3rN \) degrees of freedom forms a mode of oscillation specified by a wave vector \( \vec{q} \) and polarization \( s \) (the polarization takes \( 3r \) values).

a) Show that the canonical average at a temperature \( T \) of the quantum parameter \( n(q,s) \), the average number of phonons per mode of oscillation has the usual Bose form:

\[
n(q,s) = \frac{1}{\exp(h\omega_{q,s}/kT) - 1}
\]

b) Show that the energy of the crystal under consideration can be given as follows

\[
\sum_{q,s} \frac{1}{2} h\omega_{q,s} + \sum_{q,s} h\omega_{q,s} n(q,s)
\]

Interpret the terms here.

c) Let \( \theta \) and \( \phi \) be the angles, specifying the direction of the vector \( \vec{q} \). Taking into account that \( V/8\pi^3 \) gives the number of wave vectors per unit volume in the q space for each direction of polarization s, find the thermal energy of phonons.

d) Find the heat capacity of phonons. (Leave your result as an integral).
Mathematical Physics

1) Green’s functions

a) Consider the Helmoltz equation.

\[ (\nabla^2 + k^2)\psi(x) = f(x) \]

Write down the equation for the Green’s function to satisfy and construct the infinite one-dimensional Green’s function. The boundary conditions should be that of a wave advancing in the positive x direction.

b) Now do the same for the modified Helmoltz equation

\[ (\nabla^2 - k^2)\psi(x) = f(x) \]

Now take the boundary conditions so that the Green’s function vanishes at infinity.

2) Evaluate the following integral

\[ I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} dx \]

for 0 < a < 1.

Hint following contour may be useful
Analytical Mechanics

1) Determine the frequencies of vibration of the triatomic molecule A-B-A, as shown in the figure below. Assume that AB and BA are connected by spring of force constant K. Consider A atom’s mass \( m_\text{A} \) and B atoms mass \( m_\text{B} \). In your solution, use the Lagrangian for the system. Also assume that the atoms move only in the plane. Hint: You may need the following transformation:
\[
Q = x_1 + x_2 \\
q_1 = x_1 - x_3 \\
q_2 = y_1 + y_3
\]

2) The Lorentz force acting on a particle with charge \( q \) is given by
\[
\vec{F} = q\left\{ \vec{E} + \frac{1}{c} \dot{\vec{v}} \times \vec{B} \right\}
\]
where \( \vec{E} = -\nabla V(r) \) is the electric field of the potential \( V(r) = -\frac{k}{r} \) and \( \vec{B} = \frac{b}{r^3} \) is the external magnetic field generated by some magnetic monopole. Here, \( \dot{\vec{v}} \) is the velocity of the particle. By looking at the product \( \vec{r} \times \frac{d\vec{p}}{dt} \) show that, while mechanical angular momentum is not conserved, there is a conserved vector:
\[
\vec{D} = \vec{L} - \frac{q b}{c} \frac{\dot{r}}{r}
\]
where \( \vec{L} \) is the angular momentum.
1) The picture below shows a uniform chain of length L with just a small fraction $\varepsilon$ hanging over the edge of a frictionless table. When the chain is let go at $t=0$, it slips down over the edge
a) Find the expressions for the displacement and speed of the chain at time $t$, before it totally falls of the table.
b) Use this expressions to find the speed and acceleration just at the instant the last bit of the chain leaves the table.
Electromagnetic Theory

1) Starting from Maxwell’s equations in free space containing sources \((\rho, \vec{J})\),

a) derive the inhomogeneous wave equation satisfied by EM potentials \((\phi, \vec{A})\) using

the so-called, *velocity* gauge which is defined as: 
\[
\vec{\nabla} \cdot \vec{A} + \frac{1}{v^2} \frac{\partial \phi}{\partial t} = 0,
\]

b) What are the names for the gauges corresponding to the limits of the velocity gauge when \(v = c\) and \(v \to \infty\)?

Hint: 
\[
\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla} \left( \vec{\nabla} \cdot \vec{F} \right) - \nabla^2 \vec{F}
\]

2) Free Electron Laser – An electron beam sent at a relativistic speed \((v)\) to a static magnetic field sinusoidally varying in space with a space period (i.e., wavelength) \(\lambda_w\), gives rise to an EM wave, known as free electron lasing. Determine the wavelength (\(\lambda\)) of this free electron laser, in terms of \(\lambda_w\), \(v\) and \(c\).

Its operation principle is simple: in the rest frame of the electron beam, the static B-field is seen as an incident TEM wave which is reflected by the electron beam, much like reflection from a conducting surface.

Suggested procedure:

i) Use the fact that the phase of an EM wave - as in the term \(e^{i(k \cdot \vec{r} - \omega t)}\) - is *invariant*, and introduce a new 4-vector appears: 4-vector, \(k^\mu\). Determine its four components.

ii) Determine the transformation rules for \(k^\mu\), between two inertial frames \(S\) and \(S'\), where \(S\) moves parallel to \(\vec{k}\) with a velocity \(\vec{v}\), with respect to the original frame \(S\) in part (i).

iii) Using the previous part, identify the wavenumber, and hence the wavelength of the reflected EM wave in the rest frame of the static magnetic field.

Formula Reminder

An inertial frame moving parallel to x-axis with a velocity \(v\), with respect to a stationary frame has the following field transformations:

\[
\begin{align*}
\overline{E}_x &= E_x, \quad \overline{E}_y = \gamma \left( E_y - v B_z \right), \quad \overline{E}_z = \gamma \left( E_z + v B_y \right), \\
\overline{B}_x &= B_x, \quad \overline{B}_y = \gamma \left( B_y + \frac{v}{c^2} E_z \right), \quad \overline{B}_z = \gamma \left( B_z - \frac{v}{c^2} E_y \right),
\end{align*}
\]

\[
\Lambda = \begin{bmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \text{with} \quad \gamma = \sqrt{1 - \left( \frac{v}{c} \right)^2}, \quad \text{and} \quad \beta = \frac{v}{c}
\]
Experimental Physics

1) Draw a diagram of a Michelson interferometer. Derive an expression for and explain how to use this technique to measure the index of refraction of a given sample.
Modern Physics

1) The determination of Avogadro’s number with x-rays.

X-rays from a molybdenum target (0.626 Å) are incident on an NaCl crystal, which has the atomic arrangement shown in the Figure below. If NaCl has a density of 2.17 g/cm³ and the $n=1$ diffraction maximum from planes separated by $d$ is found at $\theta=6.41^\circ$, compute Avogadro’s number.
Quantum Mechanics

1) Consider 1D Harmonic oscillator:

a) Compute the commutators \([A^+, A^n]\) and \([A, e^{iHt}]\)

b) At \(t = 0\), a particle of mass \(m\) is in the Harmonic Oscillator state

\[
\psi(t = 0) = \frac{1}{\sqrt{2}}(u_0 + u_1)
\]

Use the Heisenberg picture to find the expected value of \(\mathcal{E}\) as a function of time. Repeat the same problem with the Schrodinger picture.

2) An operator \(f\) describing the interaction of two spin-1/2 particles has the form

\[
f = a + b\sigma_1 \cdot \sigma_2
\]

where \(a\) and \(b\) are constants, \(\sigma_1\) and \(\sigma_2\) are Pauli matrices. The total spin angular momentum is

\[
S = s_1 + s_2 = \frac{\hbar}{2}(\sigma_1 + \sigma_2).
\]

a) Show that \(f\), \(S^2\) and \(S_z\) can be simultaneously measured.

b) Derive the matrix representation for \(f\) in the \(|S, m_s, s_1, s_2>\) basis. (Do not forget to label rows and columns of your matrix.)

c) Derive the matrix representation for \(f\) in the \(|s_1, s_2, m_1, m_2>\) basis.

3) A particle has orbital angular momentum quantum number \(l = 1\) and is bound in the potential well

\[
V(r) = -V_0 \quad \text{for} \quad r < a \quad \text{and} \quad V(r) = 0 \quad \text{elsewhere.}
\]

Write down the form of the solution (in terms of known functions) in the two regions. Your solution should satisfy constraints at the origin and at infinity. Be sure to include angular dependence. Now use the boundary condition at \(r = a\) to get one equation, the solution of which will quantize the energies. Do not bother to solve the equation.
For your information:

i) The Pauli matrices are

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

ii) For 1D Harmonic oscillator lowering and raising operators are:

\[
A = \frac{\sqrt{m\omega}}{\sqrt{2}\hbar} x + i \frac{p}{\sqrt{2m\omega\hbar}}
\]

\[
A^\dagger = \frac{\sqrt{m\omega}}{\sqrt{2}\hbar} x - i \frac{p}{\sqrt{2m\omega\hbar}}
\]

iii) Radial wave equation:

\[
-\frac{\hbar^2}{2\mu} \left[ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\ell(\ell + 1)}{r^2} \right] R_{E\ell}(r) + V(r) R_{E\ell}(r) = E R_{E\ell}(r)
\]