TAKE HOME EXAM 2

TOPICS IN ALGEBRAIC GEOMETRY: ELLIPTIC CURVES

(1) Consider the cubic $E_t : (x+y+z)(xy+yz+zx) = txyz$. For which values of t is this an elliptic curve? Show that if E_t is an elliptic curve, then $E_t(\mathbb{Q})_{\mathrm{tors}}$ has a subgroup of order 6.

(2) Let Γ be a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathrm{SL}_2(\mathbb{Z}) = \bigcup \gamma_i \Gamma$ be a decomposition of $\mathrm{SL}_2(\mathbb{Z})$ into left cosets modulo Γ, where $i = 1, 2, \ldots, m$. Show that every cusp for Γ has the form $\gamma_i^{-1}(\infty) \in \mathbb{P}^1\mathbb{Q}$.

(3) Compute the cusps for $\Gamma_0(2)$. Where are the cusps located in the fundamental domain $\mathcal{F}$ (cf. lecture notes)? Which parts of the boundary of $\mathcal{F}$ are equivalent under the action of $\Gamma_0(2)$? Explain why, after gluing these parts together, you get a surface of genus 0 (with the cusps missing).

(4) Compute the rank of $E : y^2 = x^3 - 25x$ using 2-descent. Find rational points on each of the curves $C_{a,b,c}$ in $W(E)$. Find the torsion subgroup as well as a rational point in $E(\mathbb{Q}) \setminus E(\mathbb{Q})_{\mathrm{tors}}$.