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Beppo Levi and the arithmetic of elliptic curves

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1. Introduction

Most students of mathematics encounter the name of the Italian mathematician Beppo Levi in integration theory, when they learn "Beppo Levi's Lemma" on integrals of monotone sequences of functions. The attribution of this result is historically correct, but it by no means exhausts Beppo Levi's mathematical accomplishments.

Between 1897 and 1909, Beppo Levi (1875–1961) actively participated in all major new mathematical developments of the time. He was a man of great perseverance and energy, with an independent mind and a wide mathematical and philosophical culture. His list of publications includes more than 150 mathematical papers. Apart from his "Lemma", Beppo Levi is known for his work (at the very beginning of this century) on the resolution of singularities of algebraic surfaces. N. Bourbaki's *Éléments d'histoire des mathématiques* mentions Beppo Levi as one of the rare mathematicians to have recognized the Axiom of Choice as a principle used implicitly in set theory, before Zermelo formulated it.

As we shall see below, the rôle sometimes accorded to Beppo Levi in the context of set theory seems somewhat overrated. On the other hand, his truly outstanding work on the arithmetic of elliptic curves has not received the attention it deserves. He occupied himself with this subject from 1906 to 1908. His investigations, although duly reported by

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him at the 1908 *International Congress of Mathematicians* in Rome, appear to be all but forgotten. This is striking because in this work Beppo Levi anticipated explicitly, by more than 60 years, a famous conjecture made again by Andrew P. Ogg in 1970, and proved by Barry Mazur in 1976.

Shortly before his retirement, fate held in store for Beppo Levi a tremendous challenge which he more than lived up to: the last twenty years of his long life were devoted to building up mathematics in Rosario, Argentina, the place where he had emigrated.

In this article we briefly describe Beppo Levi's life and mathematical work, with special emphasis on his forgotten contributions to the arithmetic of elliptic curves. For more detailed biographical information the interested reader is referred to the extremely well researched article [Coen 1994]; a convenient list of Beppo Levi's publications is contained in [Terracini 1963, 601–606].[1] Beppo Levi's collected papers are about to be published under the direction of S. Coen by the *Unione Matematica Italiana*.

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2. Family and student years

Beppo Levi was born on May 14, 1875 in Torino, Italy, the fourth of ten children. His parents were Diamantina Pugliese and the lawyer, and author of books in law and political economics, Giulio Giacomo Levi. Perhaps the greatest mathematical talent in the family was Beppo's brother Eugenio Elia Levi, who was his junior by eight years. By the time Eugenio became a "normalista" at the elite *Scuola Normale Superiore* in Pisa, Beppo was already an active mathematician. He took great interest in the mathematical education of his younger brother and Eugenio had a brilliant career [Levi 1959–60]. In 1909 the 26 years old Eugenio was appointed professor at the university of Genova. He worked in complex analysis and the theory of Lie algebras. The "Levi condition" on the boundary of a pseudoconvex domain and the "Levi decomposition" of a Lie algebra are named after him. In World War I, Eugenio Levi volunteered for the Italian army. He died a captain, 33 years old, when the Italian army was overrun by the Austrians at Caporetto (October 21, 1917). It was the second brother Beppo lost in the war; Decio, an engineer and the last child of Beppo Levi's parents, had been killed on September 15, 1917 at Gorizia.

Beppo received his academic education at the university of his home town Torino. He enrolled in 1892, when he was 17 years old. His most influential teachers were Corrado Segre, Eugenio d'Ovidio, and also Giuseppe Peano and Vito Volterra. Although he always kept a vivid interest in all the mathematics he learned, he became most closely affiliated with Segre and thus grew up in the famous Italian school of algebraic geometry. In July 1896 he obtained his degree, the *laurea,* writing his *tesi di laurea* on the variety of secants of algebraic curves, with a view to studying singularities of space curves.

[1] See also: Homenaje a Beppo Levi, *Revista Unin Matematica Argentina* **17** (1955), as well as the special volume *Mathematicæ Notæ* **18** (1962).

3. Singularities of surfaces

While completing his *tesi di laurea*, Beppo Levi was also helping his teacher Corrado Segre proofreading Segre's important article *Sulla scomposizione dei punti singolari delle superficie algebriche.* There Segre defines the infinitely near multiple points of a singular point on an algebraic surface.

The question arose whether a certain procedure to eliminate singularities would eventually terminate. More precisely: under which conditions does the sequence of multiplicities of the infinitely near points obtained by successive quadratic transformations $x = x'z',\ y = y'z',\ z = z$, reach 1 after finitely many steps? Segre thought this was the case unless the starting point lies on a multiple component of the surface, and he wanted to deduce this from a result by the Scandinavian geometer Gustaf Kobb. But that result was not correct in its generality, and Segre's corollary was justly criticized by P. del Pezzo. It was the young Beppo Levi who supplied—with a proof that even satisfied Zariski in 1935—a complete solution of the problem which forms the content of his first publication [Levi 1897a].

So much for the mathematical substance of this particular issue between Segre and del Pezzo. But we would be depriving history of one of its spicier episodes if we did not mention the ferocious controversy between the two mathematicians of which the issue above is only a tiny detail. It is probably fortunate that most of this exchange of published notes (four by del Pezzo, the last one being entitled in ceremonial latin "*Contra Segrem*", and two by Segre) appeared in fairly obscure journals. They were not included in Segre's collected works. We understand that the exchange reflects a personal animosity predating the mathematical issues [Gario 1988], cf. [Gario 1991].

Beppo Levi continued to work for a while on the subject he had become acquainted with under these belligerent circumstances. He attacked the problem of the resolution of singularities of algebraic surfaces and claimed success.

Beppo Levi's method is that of Segre: to alternate between quadratic transformations (as above) and monoidal transformations of the ambient space, followed by generic projections. This procedure typically creates new, 'accidental' singularities of the surface. The main problem then is to control this procreation of singular points, showing that the process eventually terminates. Beppo Levi's solution was published in the paper [Levi 1897b] which appeared in the *Atti dell'Accademia delle Scienze di Torino*, Segre's house paper. It was acknowledged to be correct and complete by E. Picard in the early 1900's, by Severi in 1914, by Chisini in 1921— but *not* by Zariski in 1935. In his famous book on algebraic surfaces [Zariski 1935], Zariski points out gaps in all the proofs for the resolution of singularities of algebraic surfaces that were available at the time, including Levi's. In a dramatic climax, Zariski closes this section of his book with a note added in proof to the effect that Walker's function-theoretic proof (which had just been finished) "stands the most critical examination and settles the validity of the theorem beyond any doubt." See also the introduction to the article [Zariski 1939].

To be sure, this was not going to be the end of the history of the result.....

We will not discuss the completeness of Beppo Levi's proof. But we may quote H. Hironaka's [1962] footnote about the papers [Levi 1897a,b] from his lecture at the

International Mathematical Congress in Stockholm on his famous theorem on the resolution of singularities of arbitrary algebraic varieties in characteristic 0, a result for which he obtained the Fields Medal in Moscow in 1966: *". . . the most basic idea that underlies our inductive proof of resolution in all dimensions has its origins in B. Levi's works, or, more precisely, in the theorem of Beppo Levi which was clearly stated and given a rigorous proof by O. Zariski in his paper ...* " [Zariski 1944, section 14]. The theorem alluded to here is the main result of [1897a], which was discussed above.

Let us conclude this section with a slightly more general remark: It is commonplace today to think of the Italian geometers as a national school which contributed enormously to the development of algebraic geometry, in spite of their tendency to neglect formal precision—a tendency which is seen as the reason for the many "futile controversies" which mark this school.[2] It seems to us that there is no historical base for this point of view.

We think that a more adequate account of the period of algebraic geometry in which Beppo Levi was involved would have to measure the fundamental change of paradigm introduced in the 1930's by Zariski and Weil. For instance, trying to understand the Italian geometers, one has to try and imagine what it must have been like to think about desingularization without the algebraic concept of normalization. As for the controversies, they do not so much seem to be a result of formal incompetence, but of personal temperament and competition—examples of such feuds can be found long after the end of the Italian school of algebraic geometry. Finally, even the very name of "Italian school of algebraic geometry" can be misleading in that it does not bring out the strong European connection of this group of mathematicians.

[2] For instance D. Mumford (Parikh 1990, xxvf): "The Italian school of algebraic geometry was created in the late 19th century by a half dozen geniuses who were hugely gifted and who thought deeply and nearly always correctly about their field. ... But they found the geometric ideas much more seductive than the formal details of the proofs.... So ... they began to go astray. It was Zariski and ... Weil who set about to tame their intuition, to find the principles and techniques that could truly express the geometry while embodying the rigor without which mathematics eventually must degenerate to fantasy." — Or Dieudonné [1974, 102f]: "Malheureusement, la tendance, très répandue dans cette école, à manquer de précision dans les définitions et les démonstrations, ne tarda pas à entraîner de nombreuses controverses futiles,...."

4. Axiom of Choice and Lebesgue's theory of integration

In spite of his beautiful work on algebraic surfaces, Beppo Levi gave up his assistantship to the chair of Luigi Berzolari at the Torino University in 1899 (the year that the latter moved to Pavia), and accepted positions at secondary schools in the northern Italian towns of Vercelli, Piacenza and Torino and also in far away places like Bari and Sassari. He probably accepted these somewhat mediocre but better-paid positions in order to contribute to the finances of the family in Torino, after his father's untimely death in 1898. In 1901 Beppo Levi was candidate for a professorship (at Torino); the position was given to Gino Fano, with Beppo Levi ranking third on that occasion.

From this period dates Beppo Levi's contact with early variants of the Axiom of Choice, which earned him a mention in a footnote of Bourbaki's *Eléments d'histoire des Mathématiques* [Bourbaki 1974, p. 53]. Thanks to Moss [1979], and in particular to the extremely thorough historical study by Moore [1982, *esp.* 1.8: 'Italian Objections to Arbitrary Choices'], we may be historically a little more precise than Bourbaki (and also safely dismiss the apocryphal story told by Abraham Fraenkel in [Fraenkel, Bar-Hillel 1958, p. 48]).

Apparently following the local Torino tradition which had been started very early by G. Peano, to criticize uncontrolled application of arbitrary choices in set theory, Beppo Levi published a criticism of Felix Bernstein's thesis, pointing out a certain partition principle that Bernstein used [Levi 1902]. In this sense Beppo Levi belongs to the prehistory of Zermelo's famous article [Zermelo 1904]. Moore makes it clear, however, that Beppo Levi cannot be said to have already possessed Zermelo's Axiom of Choice. And to be sure, Beppo Levi was never ready to admit this axiom in general—see [Moore 1982, 4.7] for a discussion of Beppo Levi's later attempts to regulate the use of this and other problematic principles of set theory. Alternatively, the reader might wish to consult the easily accessible letter to Hilbert [Levi 1923] to get acquainted with Beppo Levi's peculiar idea of *deductive domains*.

A little later Beppo Levi tried to come to terms with the new theory of integration and measure of Henri Lebesgue. In a letter to Emile Borel postmarked June 1, 1906, Lebesgue writes:[3]

> *"Mon Cher Borel,*
>
> ......
>
> *Mes théorèmes invoqués par Fatou sont mis en doute actuellement par Beppo Levi dans les Rendiconti dei Lincei. Beppo Levi n'a pas su rétablir quelques raisonnements intermédiaires simples et il s'est cassé*

[3] [Lebesgue 1991, p. 148f] "My dear Borel, ... My theorems, that are being used by Fatou, are now criticized by Beppo Levi in the *Rendiconti dei Lincei.* Beppo Levi has not been able to fill in a few simple intermediate arguments and got stuck at a serious mistake of formulation which Montel once pointed out to me and which is easy to fix. Of course, I began by writing a note where I picked him out like rotten fish. But then, after a letter from Segre, and because putting down those interested in my work is not the way to build a worldwide reputation, I was less harsh....

le nez sur une faute de rédaction grave que Montel m'a jadis signalée et qu'il est facile de réparer.

Naturellement j'ai commencé par rédiger une note où je l'attrapais comme du poisson pourri puis, sur une lettre de Segre, et parce que ce n'est pas le moyen d'acquérir une réputation mondiale que d'attraper ceux qui s'occupent de mes histoires, j'ai été moins dur.

......"

Lebesgue's reply to Beppo Levi's criticism, published in the *Rendiconti dei Lincei* [Lebesgue 1906], makes it quite clear who is the master and who is the apprentice in this new field. This somewhat marginal rôle of Beppo Levi's first papers on integration may explain why his name is often lacking in French accounts of integration and measure theory. Even Dieudonné, in chapter XI (written by himself) of the historical digest [Dieudonn et al. 1978], fails to mention Beppo Levi's works altogether. In English and German speaking countries however, a course on Lebesgue's theory will usually be the unique occasion where the students hear Beppo Levi's name mentioned. His famous lemma was published in the slightly obscure *Rendiconti del Reale Istituto Lombardo di Scienze e Lettere* [Levi 1906a]. The article provides the proof of a slight generalization of one of Lebesgue's results. The statement of Beppo Levi's Lemma which we give in the inset is a resum of sections 2 and 3 of [Levi 1906a]. The lemma was quoted and thereby publicized by G. Fubini in his important paper in the *Rendiconti Acc. dei Lincei* [Fubini 1907] which contains the proof of what mathematicians still know as Fubini's theorem, in the case of a rectangle domain.

Beppo Levi's Lemma.

Let f_n be a non-decreasing sequence of integrable functions on a measurable set E such that $\lim_{n\to\infty} \int_E f_n$ is finite, then $f(x) = \lim_{n\to\infty} f_n(x)$ is finite almost everywhere and is integrable with

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

Facsimile from page 776 of [Levi 1906a]

Levi's Lemma is similar to Fatou's lemma, which coincidentally also dates back to 1906 (*Acta Math.* **30**, 335–400). It is thus possible to build up the theory without reference to Beppo Levi's result. It is nevertheless difficult to understand why Dieudonn omits Levi's name from his account of Lebesgue's theory.

Fubini's theorem for a rectangle, to quote Hawkins [1975, p. 161], "marked a real triumph for Lebesgue's ideas. As Fubini said, the Lebesgue integral 'is now necessary in this type of study'." In fact, Fubini's theorem had been anticipated by Beppo Levi, albeit without a detailed proof, in a footnote of a very substantial paper on the Dirichlet Principle [Levi 1906b, p. 322]. Here Beppo Levi observed that Pringsheim's careful investigation of double integrals in Riemann's theory of integration carries over to Lebesgue's theory, yielding in fact a simpler and more general statement.

This paper has inspired a number of works in functional analysis and variational calculus. Thus Riesz [1934] derived from its section 7 the idea for an alternative proof (not using separability) of the existence of orthogonal projections onto a closed subspace of a Hilbert space. The proof uses what other authors isolate as "Beppo Levi's inequality"—see for instance [Neumark 1959, 5.2].

Furthermore, a certain class of functions studied in [Levi 1906b] led Nikodym to define the class of what he called (BL)-functions [Nikodym 1933]. This idea was carried further in the study of so-called "spaces of Beppo Levi type" [Deny, Lions 1953].

It is also possible that a remark in [Levi 1906b] inspired some of Lebesgue's later contributions to Dirichlet's Principle. A passage in another letter of Lebesgue to Borel (12 February 1910) seems to suggest this. But the history of Dirichlet's Principle at that time is very dense and intricate, so we do not go into details here.

5. Elliptic curves

In December 1906, ten years after his *laurea*, Beppo Levi was appointed professor for *geometria proiettiva e descrittiva* at the University of Cagliari on the island of Sardinia. Because of the Italian system of *concorsi,* this somewhat isolated place has been the starting point of quite a number of academic careers. For instance, in the 1960'ies, the later Fields medallist Enrico Bombieri, Princeton, also was first appointed professor at Cagliari.

At the end of the year of 1906, Beppo Levi was a candidate for the Lobachevsky prize of the Academy of Kazan, on the basis of two papers on projective geometry and trigonometry. In spite of the positive scientific evaluation of the works he only received an "honorable mention" because, it was said, the prize was reserved for contributions to noneuclidean geometry. In fact, the prize was not awarded at all that year [Kazan 1906].

Looking back to Beppo Levi's publications discussed in the last section, and turning now to his study of cubic curves, it may be said that the year 1906 was probably the richest year for Levi's mathematical production. It was probably early that year that he began to work on the arithmetic of cubic curves.

We have seen how Beppo Levi often became acquainted with a new theory by way of a critical reading of seminal papers. Now, the very domain of the *Arithmetic of Algebraic Curves* had been defined and christened by Henri Poincar in his momentous research programme [Poincar 1901]. This programme is best understood as the attempt to reform the theoretically very unsatisfactory tradition of *diophantine analysis*. This branch of mathematics resembled recreational mathematics in that it was perfectly happy every time a certain class of diophantine equations could be solved explicitly (or shown to be unsolvable) by some trick adapted to just these equations. Now Poincar proposed to apply some of the notions developed by algebraic geometry during the nineteenth century.

More precisely, Poincar's idea was to study smooth, projective algebraic curves over the rational numbers up to birational equivalence. The first birational invariant that comes to mind is of course the *genus*, and a typical first problem studied is then the nature of the set of rational points of a curve of given genus g. Poincar starts with the case of rational curves, *i.e.* $g = 0$. Their rational points (if they exist at all) are easily parametrized and, although Poincar does not interrupt his flow of ideas by a reference, a remarkably complete study of this case had in fact been published by Hilbert and Hurwitz in 1890.

The case of curves of genus 1 with a rational point, *i.e.* of elliptic curves, occupies the bulk of Poincar's article. Without loss of generality one may assume the elliptic curve is given as a curve of degree 3 in $\mathbf{P}^2$, by a nonsingular homogeneous cubic equation. It is the theory of the rational points of these curves, as sketched by Poincar, that Beppo Levi is picking up, criticizing, and developing it further in 1906.

Any line in $\mathbf{P}^2$ meets a cubic curve E in three points (counting multiplicities). If the line and the curve are both defined over $\mathbf{Q}$, and two of the three points of intersection are rational, *i.e.* have rational homogeneous coordinates in $\mathbf{P}^2$, then so is the third. This defines a law of composition $E(\mathbf{Q}) \times E(\mathbf{Q}) \longrightarrow E(\mathbf{Q})$, called the chord and tangent method (inset). Except in trivial cases (when the set $E(\mathbf{Q})$ is too small), this does not afford a group structure on the set of rational points. But it may always be turned into an abelian group, essentially by choosing an origin. Neither Poincar nor Beppo Levi take this step towards the group structure and both work with the chord and tangent process itself. Flex points are then special, in that starting from such a point, the method does not lead to any new point.

Picture with chord and tangent method and remark on points of inflection.

In a way, this basic method of the arithmetic of elliptic curves had been used by Fermat when working with certain diophantine problems, and in particular in some of his proofs by *infinite descent*. But the geometric meaning of it seems to have first been observed by Newton—see [Schappacher 1990] for a more detailed history of the method.

In 1906–1908 Levi published four remarkable papers on the subject in the *Atti della Reale Accademia delle Scienze di Torino* [Levi 1906–08]. The first paper is rather general. Beppo Levi avoids the difficult question as to whether a plane cubic curve possesses a rational point or not, by assuming, once and for all, that the curves under consideration have at least one rational point. He classifies these elliptic curves up to isomorphism, not only over $\mathbf{C}$, but over $\mathbf{Q}$.

Generalizing the chord and tangent process, Beppo Levi also considers deducing new rational points on E from given ones by intersecting E with curves defined over $\mathbf{Q}$ of degree higher than 1. He knows, as did Sylvester and others before him, that this apparently more general notion of *rational deduction of points* does not yield any more general dependencies than the chord and tangent method.

In the first paper Beppo Levi gives in particular a birationally invariant definition of the *rank* with respect to the chord and tangent process of the set of rational points on an elliptic curve over $\mathbf{Q}$, under an assumption which amounts to saying that the group $E(\mathbf{Q})$

is a finitely generated abelian group. He justly criticizes Poincar for having overlooked that, given two birationally equivalent curves, one may and the other may not have a rational point of inflection—this actually makes Poincar's notion of rank not birationally invariant!

Beppo Levi's notion of rank does not coincide with what we call today the **Z**-rank of the finitely generated abelian group $E(\mathbf{Q})$: Beppo Levi adds to the free rank the minimum number of points needed to generate the torsion subgroup.[4]

In a footnote he stresses very explicitly that the assumption of finite generation for $E(\mathbf{Q})$ was not proved (it was established only in 1922, by J.L. Mordell):[5]

> *"Può cioè dubitarsi, sia che esista sulla cubica una base costituita di infiniti punti razionali: si potrebbe dire allora che il rango è infinito; sia che non esista alcuna base in quanto che, fissato un qualunque gruppo di punti razionali, i suoi punti si possano dedurre razionalmente da altri senza che questi si ottengano razionalmente dal gruppo fissato: è ciò che avverrebbe se ogni punto razionale fosse tangenziale d'altri punti razionali."*

Thus Beppo Levi is more explicit than Henri Poincaré, who did not let the possibility of $E(\mathbf{Q})$ not being finitely generated enter into his discussion. Like Beppo Levi's footnote, Mordell's proof of 1922 has two parts: it is shown that the rank cannot be infinite and then, by means of the theory of heights, it is shown that the second possibility indicated by Levi, does, in fact, not occur.

The last part of Beppo Levi's first note is devoted to elliptic curves all of whose points of order 2 are rational. For these curves Beppo Levi seems to embark upon a general 2-descent, but he does not quite conclude it.

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[4] To be precise, the minimality condition that Beppo Levi writes down for his (finite) *basis* of the set of rational points is not strong enough to make the rank uniquely defined; he only asks that a basis be minimal in the sense that none of its points be expressable in terms of the others.

[5] "[The finite rank assumption] may be doubtful: either there might exist a cubic curve with a basis consisting of infinitely many rational points: in this case one would say that the rank is infinite; or no basis exists at all in the sense that, for any given set of rational points, one can obtain these points rationally from other points which themselves cannot be obtained rationally from the given set: this would occur if every rational point were on the tangent of another rational point."

6. "Ogg's conjecture"

Starting from a given rational point, other rational points on a given elliptic curve may be constructed applying successively the chord and tangent method, but only to the given point or to points constructed in previous steps. As Levi puts it [Levi 1906–08, 11], usually one would in this way obtain infinitely many rational points, but in certain exceptional cases the procedure 'fails' in the sense that one ends up in some kind of a loop and obtains only finitely many points. In modern language this means that the point of departure has finite order in the group $E(\mathbf{Q})$.

Beppo Levi sets out to classify these 'failures' of the chord and tangent method. In modern terms: he wishes to determine what the structure of the subgroup of the points of finite order on an elliptic curve over $\mathbf{Q}$ can be. The last three papers in the series are devoted to this problem.

Beppo Levi's method is straightforward. He takes a general non-singular cubic curve and writes down explicitly what the 'failure' of the chord and tangent method for a given rational point on the curve means for the coefficients of the equation. The chord and tangent method can of course 'fail' in various ways and each way gives rise to a certain finite configuration of points and lines. Levi distinguishes four types: *configurazioni arborescenti*, *poligonali* and *poligonali misti* and finally *configurazioni con punti accidentali*.

Let us translate this into modern terminology: Levi fixes a finite abelian group A and computes under which conditions on the coefficients of the curve, the group of rational points admits A as a subgroup. He is perfectly aware of the complex analytic theory of elliptic functions and he exploits this theory as well as the restrictions on the structure of A that come from the fact that the elliptic curve is already defined over $\mathbf{R}$ (in fact, it is even defined over $\mathbf{Q}$): if A occurs, it is either a cyclic group, or a cyclic group times $\mathbf{Z}/2\mathbf{Z}$. Beppo Levi's *configurazioni arborescenti*, *poligonali*, and *poligonali misti* correspond respectively to subgroups of the form $A = \mathbf{Z}/n\mathbf{Z}$ where n is a power of 2, n is odd, or n is even but not a power of 2. The *configurazioni con punti accidentali* correspond to the groups $A = \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, with n even.

Beppo Levi's method yields explicit parametrizations of elliptic curves with given torsion points; the problem of their existence then comes down to solving certain diophantine equations. Sometimes this is very easy. Thus Levi shows that the groups

$$\begin{array}{rl} \mathbf{Z}/n\mathbf{Z} & \text{for } n = 1, 2, \ldots, 10 \text{ and } 12, \\ \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} & \text{for } n = 2,\ 4,\ 6 \text{ and } 8. \end{array}$$

all occur infinitely often.

What is more remarkable: Beppo Levi can also show that certain *configurazioni* **do not occur**: the group $A = \mathbf{Z}/n\mathbf{Z}$ does not occur for $n = 14$, 16 and $n = 20$ and $A = \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ does not occur for $n = 10$ and $n = 12$. In these cases he must study some thorny diophantine equations, defining plane curves of genus 1 or 2. He concludes by infinite descent, very much in the spirit of Fermat. In the jargon of the contemporary arithmetic of elliptic curves the infinite descent involves a 2-descent [Levi 1906–08, 17].

In some other cases Beppo Levi does not overcome the technical difficulties; for $n = 11$ and $n = 24$ he gives the equations but cannot rule out the existence of unexpected solutions.

The equations Levi finds—say, for the groups $\mathbf{Z}/n\mathbf{Z}$—are equations for the modular curves $X_1(n)$ that parametrize elliptic curves together with a point of order n. The modular curves called today $X_0(N)$ and $X(N)$ and their explicit equations—which we see as fairly similar to $X_1(n)$—had been studied in the 19th century, see for instance [Kiepert 1888–90]. There is no indication that Beppo Levi was aware of this connection. He seems to have looked at his equations only the way he obtained them: parametrizing families of elliptic curves with given torsion points. Neither upper half plane, modular groups, nor modular functions are evident in his work.

The "easy" cases, where a parametrization was found, are precisely the cases where the genus of $X_1(n)$ is zero. The equation that Beppo Levi finds for $n = 11$ is, what we recognize today as a beautiful $\mathbf{Z}$-minimal equation for the curve $X_1(11)$ of genus 1:

$$Y^2X - Y^2Z - X^2Z + YZ^2 = 0.$$

Its five obvious rational points are all "cusps" of $X_1(11)$. Beppo Levi does not associate the complex analytic picture of the cusps with them, but he observes that they correspond to degenerate curves.

At the 1908 International Mathematical Congress in Rome, Beppo Levi reports on his work on elliptic curves [Levi 1909]. There he also says what he thinks happens for the other values of n: he believes that the above list exhausts all possibilities.

This is the way he states it: for the *configurazioni arborescente* he has proved that $\mathbf{Z}/n\mathbf{Z}$ where n is a power of 2 cannot occur for $n = 16$ and therefore n *"non può mai possedere il fattore 2 a potenza superiore alla* 3^a*."*[6] For the *configurazioni poligonali* he writes regarding the group $\mathbf{Z}/n\mathbf{Z}$ with n odd: *"È molto probabile che per* $n > 9$ *non esistano più configurazioni di punti razionali ..."*[7] As far as the *configurazioni poligonali misti* are concerned, he remarks that he has showed that $\mathbf{Z}/n\mathbf{Z}$ cannot occur when $n = 20$ and that therefore $n = 5 \cdot 2^k$ cannot occur for any $k \geq 2$. He has shown that $\mathbf{Z}/2n\mathbf{Z}$ does not occur for $n = 7$ and *"è probabile che non esistano nemmeno per maggiori fattori dispari del* n*."*[8]

It is not difficult to see that these conjectures already imply that the above list for the groups of the type $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ should be complete: *"Esistono tali configurazioni per* $n = 2, 4, 8, 6$, *ma non esistono per* $n = 12$ *e per* $n = 10$ *ed è da arguire che non ne esistano per valori maggiori di* n*."*[9]

Apart from the group $\mathbf{Z}/24\mathbf{Z}$ which he does not mention or forgets to mention, this means precisely that Beppo Levi believes that the list of groups above is complete. More than 40 years later T. Nagell formulates the same conjecture [Nagell 1952a].[10] In our days the conjecture became widely known as Ogg's conjecture, after Andrew Ogg, who formulated it 60 years after Beppo Levi.

[6] ... cannot contain the factor 2 to a power exceeding 3.

[7] It is very probable that for n greater than 9 there do not exist any more rational points ...

[8] it is probable that they do not exist either for even larger odd values of n.

[9] Such configurations exist for $n = 2, 4, 8, 6$, but not for $n = 12$ and 10; one can argue that they do not exist for larger values of n.

[10] We thank Prof. A. Schinzel for bringing Nagell's paper to our attention.

The problem studied by Beppo Levi and later by Billing, Mahler, Nagell and Ogg has been very important in the development of arithmetic algebraic geometry. In the years following 1970 rapid progress was made, by invoking the arithmetic theory of modular curves. The cases $n = 11, 15, 24$ had already been taken care of before the fifties [Billing and K. Mahler 1940], [Nagell 1952b]. It is touching to read Beppo Levi's review of [Billing and K. Mahler 1940] in the *Mathematical Reviews* where he recognizes the equation of $X_1(11)$ that he had already published in 1908 but was unable to solve completely at the time.

Ogg showed that $n = 17$ does not occur in his important paper [Ogg 1971] where the connection between this problem and the theory of modular curves is spelled out. Ligozat [1975] and Kubert [1976] take care of several small values of n. In 1973 Mazur and Tate [1973] show that there do not exist rational points of order $n = 13$ on elliptic curves over $\mathbf{Q}$. This exceptional case is eliminated by means of a 19-descent (!), performed in the language of flat cohomology, on a curve of genus 2. Finally in 1976, Barry Mazur proves the conjecture. It is a consequence of his careful study of the modular curves $X_0(n)$ which are closely related to the curves $X_1(n)$. His paper [Mazur 1977] is a milestone in arithmetic algebraic geometry. The techniques developed therein are basic in the proof of the main conjecture in Iwasawa theory by Mazur and Wiles [1984] in 1980 and in Ribet's result [Ribet 1990] reducing Fermat's Last Theorem to the Conjecture of Taniyama, Shimura and Weil. The latest development concerning torsion on elliptic curves is Loïc Merel's proof (Spring 1994) of the general boundedness conjecture. This is the statement that the K-rational torsion of any elliptic curve defined over a field K of degree d over $\mathbf{Q}$ is bounded in terms of d alone. Beppo Levi's conjecture is an explicit version of the special case $d = 1$ of this new theorem. Merel's proof builds upon Mazur's work and subsequent refinements by S. Kamienny, combining them with other recent results in the arithmetic of elliptic curves.

There is no elliptic curve over Q with a rational 16-torsion point.

d'aprs Beppo Levi

Beppo Levi uses the curves given by the equations

$$Y^2(X - Z) - aX^2Y + (a + b)XYZ - bXZ^2, \qquad (a, b \in \mathbf{Q})$$

in $\mathbf{P}^2$. They possess the rational points $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 0 : 1)$, $P_3 = (0 : 1 : 0)$ and $P_4 = (1 : 1 : 1)$. Each point is on the tangent of the previous one. The point P_1 has order 8 if we take P_4 as the neutral element of the group of points, i.e., if P_4 is a point of inflection. This means that the Hessian vanishes in P_4 and this is equivalent to

$$b(a^2 - 3a + 2 + b) = 0.$$

The case $b = 0$ corresponds to a degenerate cubic, but all other curves with $b = -a^2 + 3a - 2 = -(a - 1)(a - 2)$ are smooth curves containing the rational point P_1 of order 8. Conversely, every elliptic curve over $\mathbf{Q}$ with a point of order 8 is of the above form for some $a, b \in \mathbf{Q}$.

Before studying rational points of order 16, we observe that not only does the tangent of P_1 intersect the cubic in P_2, but so does the tangent of the point $P_1' = (1 : -a^2 - 2a : a)$. In terms of the group law we have that $P_1' = 5P_1$.

Suppose that Q is a point on the curve of order 16. Without loss of generality we may assume that $-2Q = P_1$, i.e., Q is a point which has P_1 on its tangent. There are three more such points: Q', Q'' and Q'''. They are not necessarily rational, but if Q has rational coordinates, so does at least one other point, say, Q'. Let l_1 be the line joining Q and Q' and let l_2 be the line joining Q'' and Q'''. Beppo Levi views $l_1 \cup l_2$ as a degenerate member of the *fascio* $\mathfrak{F}$ of conics through Q, Q', Q'' and Q'''. He explicitly determines $\mathfrak{F}$ by observing that the partial derivative of the cubic equation with respect to X decribes a conic in $\mathfrak{F}$ and so does the equation $YZ = aX^2$ (subtract X times the equation of the first conic from the cubic equation and divide by Y). The conic $l_1 \cup l_2$ is the unique member that passes through P_1'. It is not difficult to compute the equations of l_1 and l_2:

$$Y - a(1 \pm \sqrt{a-1})X \pm \sqrt{a-1}Z = 1 - a.$$

If Q is a rational point, the lines l_1 and l_2 admit equations with rational coefficients, i.e.,

$$a - 1 = c^2 \qquad \text{for some } c \in \mathbf{Q}.$$

The line l_1 intersects the cubic in the points P_1', Q and Q'. The coordinates of $Q = (x : y : z)$ satisfy the equation

$$(c^2+1)x^2 + (c^2+1)(c-1)xz + c(c-1)z^2 = 0.$$

If the point Q is rational, the discriminant of this quadratic equation is a square:

$$d^2 = (c+1)(c-1)(c^2+1)(c^2-2c-1) \qquad \text{for some } d \in \mathbf{Q}.$$

Beppo Levi solves this equation by means of Fermat's method of infinite descent. There are two possibilities. Up to sign, either $c-1$, $c+1$, c^2+1 and c^2-2c-1 are all squares in $\mathbf{Q}$ or they are all 2 times a square. In the first case one easily arrives at the equation

$$X^4 + 1 = 2Y^2$$

and in the second case at

$$X^4 + 1 = Y^2.$$

Already in 1738 Euler had shown that the first equation only admits the trivial rational solutions $(X, Y) = (\pm1, \pm1)$ and $(0, \pm1)$. The analogous fact for the second equation had been known already to Fermat: Its solutions correspond to $(c, d) = (0, \pm1)$ and $(\pm1, 0)$. This implies $b = 0$ which yields a degenerate cubic curve. These points together with the two points at infinity therefore account for the six rational cusps of the curve $X_1(16)$.

We conclude that an elliptic curve over $\mathbf{Q}$ cannot have any rational points of order 16.

Picture of configurazioni

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7. Parma — Bologna — Rosario

Having treated the first ten years of Beppo Levi's professional life rather extensively, we will be much shorter on the remaining fifty (!) years. This half century from 1908 to the end of the 1950'ies falls naturally into three periods: almost 20 years in Parma, 10 at Bologna, and a good 20 years in Argentina.

In 1909 Beppo Levi married Albina Bachi. She was from the town of Torre Pelice in Piemonte, the alpine north west of Italy, as Beppo Levi. He had started visiting his future in-laws in 1906, the year of his nomination at Cagliari. Beppo and Albina had three children, Giulio, Laura and Emilia. At the end of 1910 the family left Cagliari: Beppo Levi was appointed at the university of Parma, on Italy's mainland. He stayed there until 1928. Among his uninterrupted production (increasingly also on questions of mathematics teaching), there is one remarkable number-theoretic attempt from this Parma period: in 1911 he published a paper on the geometry of numbers in the *Rendiconti del Circolo Matematico di Palermo* [Levi 1911] where he claimed to give a proof of a conjecture of Minkowski's concerning *critical* lattices in $\mathbf{R}^n$. However, Levi's proof appears to be incomplete; see [Keller 1930] who mentions a letter of Beppo Levi in which he acknowledged a gap in his proof. A complete proof of this result was given only in the 1940'ies [Hajós 1942].

Thus it was in Parma that the Levis lived through World War I and the ensuing political transformation of Italy, and of Europe. A reflection of these events—immediately painful for Beppo Levi through the death of two of his younger brothers—can be found in his speech (11 January 1919) at Parma University for the opening of the academic year 1918–1919, on "Nations and Humanity" [Levi 1919].

In the 1920's, the first decade of Mussolini's rule, Beppo Levi was antifascist. He signed the Croce manifesto in 1925. Around that time his situation at Parma University became increasingly difficult because more and more disciplines had to be suppressed for budget reasons. In the end, the sciences were reduced to chemistry and Levi was the only professor left at the mathematics department. It therefore came as a great relief when he obtained his transfer to Bologna—a town with a traditionally famous university—at the end of 1928, after all obstacles to his nomination there had finally been overcome.

While in Bologna he held various posts in the Italian Mathematical Society (U.M.I.), and took care of the *Bolletino dell'Unione Matematica Italiana* for many years. It was through correspondence related to a paper submitted to this journal that Beppo Levi first entered into contact with a mathematician from Argentinia.

In spite of his personal critical attitude to fascism, Beppo Levi took the oath to fascism in 1931, like most other Italian mathematicians. In fact, this oath was generally considered a mere formality; even the church held that it was a legitimate claim by the government for obedience. The mathematician Levi-Cività added a private reservation, and the government showed that it was quite prepared to accept the substance without the form. Of roughly 1200 professors only 11 refused to sign. The 71 years old Vito Volterra was one of them. Volterra, by the way, stayed in Italy, where he died in 1940—so the SS-car that came to his house in 1943 to deport him had to leave empty.....

It was only after the rapprochement between Hitler and Mussolini, in 1938, that Italian fascism adopted some of the violent racial policies of the Nazis which at that time were preparing for their monstrous climax in Germany and German-controlled Europe. Thus Levi-Cività, as well as Beppo Levi and a total of 90 Italian jewish scholars lost their jobs in 1938, and most of them had to start looking for a country of refuge.

Beppo Levi was 63 years old when he lost his professorship in Bologna. At age 64 he started as the director of the newly-created mathematical institute at the *Universidad del Litoral* in Rosario, Argentina.

The founding of this institute at Rosario, upstream from Buenos Aires,[11] took place at a time of cultural expansion of several Argentinian cities upcountry, mainly Rosario, Crdoba and Tucumn. A relative prosperity helped in the development of more substantial groups of professionals, mainly lawyers, medical doctors and engineers, who prometed local cultural activity in these cities and invited leading intellectuals and artists from Buenos Aires to lecture or visit there. These professionals were financially better off and also had started to mix with people in an even better financial situation who were their clients. Societies, orchestras, art galleries and also publishing houses began to emerge in this period in Rosario.

The official opening ceremony of the mathematical institute in Rosario was held in 1940. Lectures were delivered by Corts Pl, Rey Pastor and Beppo Levi. These two Argentinians had been the key to Beppo Levi's arrival in Argentina. Pl was an engineer who taught physics and had an active interest in the history of science. He was a friend and admirer of Rey Pastor, the undisputed leading mathematician of Argentina at the time.

Beppo Levi was extremely active in Rosario. Apart from organizing and managing the Institute (assisted by Luis A. Santal) he founded and edited a journal and a book series of his institute. The journal appeared for the first time already in 1939, and as of 1941 was called *Mathematicæ Notæ* (Boletin del Instituto de Matematica). Roughly one third of Beppo Levi's publications are in Spanish. These are his papers from the Argentinian period, many of which appeared in the *Math. Notæ*. Beppo Levi continued teaching at Rosario until the age of 84.

In 1956, shortly before he turned 81, he was awarded the Italian *Premio Feltrinelli*. Unfortunately the official text of the prize committee [Segre 1956] shows a somewhat insecure appreciation of some of Beppo Levi's works. At the end of the evaluation the committee of this prize for Italian citizens congratulates itself that Beppo Levi has highly honoured the name of Italy by his working in Argentina ...

Beppo Levi died on 28 August 1961, 86 years old, in Rosario where his institute is now named after him. He was probably the shortest mathematician in our century, with the longest professional activity.

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[11] We are grateful to Eduardo L. Ortiz, London, for the information on Argentina contained in the following paragraphs. For the opening of the Rosario institute, see *Publicaciones del Instituto de Matemticas de la Universidad Nacional del Litoral* $\mathbf{3_5}$ (1940), cf. E. Ortiz (ed.), *The works of Julio Rey Pastor*, London 1988.

Pictures of Beppo Levi (at least 3), Mazur, Ogg, Lebesgue, Corrado Segre, the institute in Rosario. E.E. Levi. Math. Notæ.

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