HOMEWORK 2

TOPICS IN AG: ELLIPTIC CURVES

Due March 03, 2004

(1) Compute the tangents to the hyperbola $X^2 - Y^2 = 1$ and to the parabola $Y = X^2$ (over the real numbers) at their points at infinity. Use the insight gained to give a definition of the concept of an asymptote for algebraic curves defined over arbitrary (e.g. finite) fields.

(2) Let K be a field of characteristic $\neq 2$, and $f \in K[X]$ a polynomial of degree ≥ 4 without multiple roots. Show that the projective closure of the hyperelliptic curve $y^2 = f(x)$ has exactly one singular point.

(3) Let $f, g, h \in K[x, y]$ be polynomials, and put $f = gh$. Show that any point oof intersection of the curves $g(x, y) = 0$ and $h(x, y) = 0$ is a singular point of the curve $f(x, y) = 0$.

(4) Show that the Klein quartic

$$X^3Y + Y^3Z + Z^3X = 0$$

defined over a field K is smooth if and only if K has characteristic $\neq 7$.

(5) Determine the number of points at infinity of the projective closure of the unit circle $x^2 + y^2 = 1$ over the finite fields $\mathbb{F}_3$, $\mathbb{F}_5$ and $\mathbb{F}_9$.

(6) Consider the parabola $\mathcal{C} : y = x^2$ over some ring R. Show that the geometric group law defined for conics specializes to

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3) \quad \text{for } x_3 = x_1 + x_2.$$

Deduce that $\mathcal{C}(R) \simeq (R, +)$, the additive group of R.

(7) Consider the hyperbola $\mathcal{C} : xy = 1$ over some ring R. Show that the geometric group law defined for conics specializes to

$$(x_1, y_1) + (x_2 b y_2) = (x_3, y_3) \quad \text{for } x_3 = x_1 x_2.$$

Deduce that $\mathcal{C}(R) \simeq R^\times$, the unit group of R.