ELEMENTARY NUMBER THEORY

MORE PROBLEMS

(1) Give an example of non-unique factorization in the monoid
$$M = \{1, 5, 9, 13, \ldots\}$$
of natural numbers of the form $4n + 1$, and explain why your example is correct.

(2) Consider the statement
$$x^2 \equiv 1 \bmod n \Longrightarrow x \equiv \pm 1 \bmod n. \qquad (*)$$
(a) Show this is true if $n = 2p$ is twice a prime number.
(b) Is $(*)$ true for $n = 9$? More generally, does it hold if $n = p^2$ is the square of a prime?
(c) Give an example that shows $(*)$ is not true if $n = 3p$ is three times a prime number $p > 3$.

(3) Show that there are infinitely many primes of the form $p \equiv \pm 2 \bmod 5$.

(4) Let $n \equiv 7 \bmod 8$ be a positive integer. Show that n cannot be written as a sum of three squares.

(5) Show that $x^2 + y^2 = 3z^2$ does not have a nontrivial solution in integers. Hint: reduce modulo something.

(6) Let $p > 3$ be a prime, and assume that $p = e^2 + 3f^2$ for integers e and f. Show that $(-3)^{(p-1)/2} \equiv 1 \bmod p$ for all such primes.

(7) Let p be an odd prime number. Find all natural numbers $x, y \in \mathbb{N}$ with $x^3 + y^3 = p$.

(8) Let p be an odd prime number. Find all natural numbers $x, y \in \mathbb{N}$ with $x^3 - y^3 = p$ [This is not easy. Proceed as far as you can; try a few explicit values for p].