MIDTERM 2

LINEAR ALGEBRA 2004

(1) Let P_4 be the space of polynomials of degree ≤ 4. Prove that
$$(p,q) = \int_0^2 (t-1)^2 p(t)q(t)\,dt$$
is an inner product and, given this inner product, find the kernel of the linear transformation $\mathrm{pr}_W\colon P_4 \to W$, where $W = P_2 \subset P_4$.

(2) The inner product on $\mathbb{R}^4$ is given by $(a,b) = 2a_1b_1 + a_2b_2 + a_3b_3 + 2a_4b_4$. Use the Gram-Schmidt process to find an orthonormal basis in $W = \mathrm{Span}\{u_1, u_2, u_3\}$, where
$$u_1 = \begin{bmatrix} -2 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} -1 \\ 4 \\ 5 \\ 3 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} -5 \\ 5 \\ 10 \\ 11 \end{bmatrix}.$$

(3) Prove that for any two $(m \times n)$-matrices A and B one has $\mathrm{Tr}(A^{\mathrm{T}}B) = \mathrm{Tr}(B^{\mathrm{T}}A)$.

(4) Find the polynomial $p(t)$ of degree ≤ 3 such that $p(-1) = p(1) = 0$ and the value of
$$\int_{-1}^{1} \left|p(t) + 3t - 5\right|^2 dt$$
is minimal possible.

(5) Let
$$C = \begin{bmatrix} 3 & a \\ a & 2 \end{bmatrix}.$$
Find the values of a for which the function $(x,y) = x^{\mathrm{T}}Cy$ is an inner product on $\mathbb{R}^2$.

Hints

(1) Recall that projections onto subspaces are defined by writing the vector space as a direct sum of subspaces. Here we have to consider the decomposition $P_4 = P_2 \oplus (P_2)^\perp$. We know that the kernel of the projection onto the first component P_2 is the second component $(P_2)^\perp$, so we will be done once we have computed the orthogonal complement of P_2.

Now $(P_2)^\perp$ consists of all polynomials p of degree ≤ 4 with $(p, 1) = (p, t) = (p, t^2) = 0$.

(2) Later. I have a horrible headache tonight.

(3) Obviously we have $\operatorname{Tr} C = \operatorname{Tr} C^{\mathrm{T}}$ for all square matrices C. Can you use this somehow?

(4) Write $p(t) = a + bt + ct^2 + dt^3$. Then $p(-1) = p(1) = 0$ give $c = -a$ and $d = -b$, i.e. $p(t) = a + bt - at^2 - bt^3$.

If you do the calculations, you will find that

$$\int_{-1}^{1} (p(t) + 3t - 5)^2 dt = \frac{16}{15}a^2 - \frac{40}{3}a + \frac{16}{105}b^2 + \frac{8}{5}b + 56.$$

This function becomes extremal for $a = \frac{25}{4}$ and $b = -\frac{21}{4}$, and the minimum is $\frac{152}{15}$.

Where's the linear algebra? I have no idea.

(5) Bilinearity (i.e. the properties $(x + x', y) = (x, y) + (x', y)$ and $(cx, y) = c(x, y)$ as well as symmetry are easily checked and hold for arbitrary values of a. Thus we have an inner product if and only if $(x, x) \ge 0$ with equality if and only if $x = 0$. This should not be too hard if you remember how to complete squares. The answer is: whenever $0 < a^2 < 6$.