LINEAR ALGEBRA

LAST YEAR'S FIRST MIDTERM

2. Find a basis for the space $V \subset P_5$ of polynomials p of degree up to 5 such that

$$p(2) - \frac{dp}{dt}(2) = \frac{d^2p}{dt^2}(2) = \frac{d^3p}{dt^3}(2) = 0.$$

What is the dimension of this space?

V consists of polynomials $p(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ satisfying

$$\begin{aligned} 32a_5 + 16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0 - 80a_5 - 32a_4 - 12a_3 - 4a_2 - a_1 &= 0, \\ 160a_5 + 48a_4 + 12a_3 + 2a_2 &= 0, \\ 240a_5 + 48a_4 + 6a_3 &= 0. \end{aligned}$$

Simplifying a bit this gives

$$\begin{aligned} 48a_5 + 16a_4 + 4a_3 - a_1 - a_0 &= 0, \\ 80a_5 + 24a_4 + 6a_3 + a_2 &= 0, \\ 40a_5 + 8a_4 + a_3 &= 0. \end{aligned}$$

More simplification:

$$\begin{aligned} 8a_5 - a_3 - a_2 - a_1 - a_0 &= 0, \\ 8a_4 - 7a_2 - 10a_1 - 10a_0 &= 0, \\ 2a_3 + 4a_2 + 5a_1 + 5a_0 &= 0. \end{aligned}$$

At this point we see that we can choose $a_0 = t$, $a_1 = s$ and $a_2 = r$ freely; then a_3, a_4 and a_5 can be computed from the third, second, and first equation. In fact we get

$$\begin{aligned} a_0 &= t & a_3 &= \tfrac{1}{2}(-4r - 5s - 5t) \\ a_1 &= s & a_4 &= \tfrac{1}{8}(7r + 10s + 10t) \\ a_2 &= r & a_5 &= \tfrac{1}{16}(-2r - 3s - 3t) \end{aligned}$$

Thus the polynomials in V have the form

$$\frac{1}{16}(-2r - 3s - 3t)x^5 + \frac{1}{8}(7r + 10s + 10t)x^4 + \frac{1}{2}(-4r - 5s - 5t)x^3 + rx^2 + sx + t.$$

A basis is given by $\{p_1, p_2, p_3\}$ with

$$\begin{aligned} p_1 &= -\tfrac{1}{8}x^5 + \tfrac{7}{8}x^4 - 2x^3 + x^2, \\ p_2 &= -\tfrac{3}{16}x^5 + \tfrac{5}{4}x^4 - \tfrac{5}{2}x^3 + x, \\ p_3 &= -\tfrac{3}{16}x^5 + \tfrac{5}{4}x^4 - \tfrac{5}{2}x^3 + 1. \end{aligned}$$

If you want, you can replace p_3 by $p_3 - p_2 = -x + 1$.

4. Find the dimension and a basis for the subspace $\operatorname{span} S \subset M_{2,2}$, where

$$S = \left\{ \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix} \right\}.$$

Since the first and the third matrices are equal, we can omit the third and find that $\operatorname{span} S = \operatorname{span} T$ for

$$T = \left\{ \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix} \right\}.$$

In order to test whether these matrices are linearly independent, we solve the system

$$a \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} + b \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} + c \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix} + d \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix} = 0,$$

where the 0 on the right hand side is the 2×2 zero matrix. This is a system of four equations in four unknowns that we can write more traditionally in the following form:

$$\left(\begin{array}{cccc|c} 3 & 4 & 5 & 0 & 0 \\ 2 & 0 & 6 & 4 & 0 \\ -1 & 0 & -3 & -2 & 0 \\ 2 & 2 & 2 & -1 & 0 \end{array} \right).$$

Solving this in the usual way we find that $a = 2$, $b = 1$, $c = -2$, $d = 2$ is a nontrivial solution. Since all the coefficients are nonzero, we can omit any of the 4 matrices and find e.g. that $\operatorname{span} S = \operatorname{span} U$ for

$$U = \left\{ \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix} \right\}.$$

Now it is easily checked that these three matrices are linearly independent, hence form a basis of $\operatorname{span} S$.