2. Find a basis for the space $V \subset P_5$ of polynomials $p$ of degree up to 5 such that

$$p(2) - \frac{dp}{dt}(2) = \frac{d^2p}{dt^2}(2) = \frac{d^3p}{dt^3}(2) = 0.$$ 

What is the dimension of this space?

$V$ consists of polynomials $p(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ satisfying

$$32a_5 + 16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0 - 80a_5 - 32a_4 - 12a_3 - 4a_2 - a_1 = 0,$$

$$160a_5 + 48a_4 + 12a_3 + 2a_2 = 0,$$

$$240a_5 + 48a_4 + 6a_3 = 0.$$ 

Simplifying a bit this gives

$$48a_5 + 16a_4 + 4a_3 - a_1 - a_0 = 0,$$

$$80a_5 + 24a_4 + 6a_3 + a_2 = 0,$$

$$40a_5 + 8a_4 + a_3 = 0.$$ 

More simplification:

$$8a_5 - a_3 - a_2 - a_1 - a_0 = 0,$$

$$8a_4 - 7a_2 - 10a_1 - 10a_0 = 0,$$

$$2a_3 + 4a_2 + 5a_1 + 5a_0 = 0.$$ 

At this point we see that we can choose $a_0 = t$, $a_1 = s$ and $a_2 = r$ freely; then $a_3$, $a_4$ and $a_5$ can be computed from the third, second, and first equation. In fact we get

$$a_0 = t \quad a_3 = \frac{1}{16}(-4r - 5s - 5t)$$

$$a_1 = s \quad a_4 = \frac{1}{8}(7r + 10s + 10t)$$

$$a_2 = r \quad a_5 = \frac{1}{16}(-2r - 3s - 3t)$$

Thus the polynomials in $V$ have the form

$$\frac{1}{16}(-2r - 3s - 3t)x^5 + \frac{1}{8}(7r + 10s + 10t)x^4 + \frac{1}{2}(-4r - 5s - 5t)x^3 + rx^2 + sx + t.$$ 

A basis is given by $\{p_1, p_2, p_3\}$ with

$$p_1 = -\frac{1}{2}x^5 + \frac{7}{8}x^4 - 2x^3 + x^2,$$

$$p_2 = -\frac{3}{16}x^5 + \frac{5}{4}x^4 - \frac{5}{2}x^3 + x,$$

$$p_3 = -\frac{1}{16}x^5 + \frac{5}{8}x^4 - \frac{5}{2}x^3 + 1.$$ 

If you want, you can replace $p_3$ by $p_3 - p_2 = -x + 1.
4. Find the dimension and a basis for the subspace \( \text{span} \, S \subset M_{2,2} \), where

\[
S = \left\{ \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix} \right\}.
\]

Since the first and the third matrices are equal, we can omit the third and find that \( \text{span} \, S = \text{span} \, T \) for

\[
T = \left\{ \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix} \right\}.
\]

In order to test whether these matrices are linearly independent, we solve the system

\[
a \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} + b \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} + c \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix} + d \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix} = 0,
\]

where the 0 on the right hand side is the \( 2 \times 2 \) zero matrix. This is a system of four equations in four unknowns that we can write more traditionally in the following form:

\[
\begin{pmatrix}
3 & 4 & 5 & 0 \\
2 & 0 & 6 & 4 \\
-1 & 0 & -3 & -2 \\
2 & 2 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Solving this in the usual way we find that \( a = 2, b = 1, c = -2, d = 2 \) is a nontrivial solution. Since all the coefficients are nonzero, we can omit any of the 4 matrices and find e.g. that \( \text{span} \, S = \text{span} \, U \) for

\[
U = \left\{ \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -2 & -1 \end{bmatrix} \right\}.
\]

Now it is easily checked that these three matrices are linearly independent, hence form a basis of \( \text{span} \, S \).