Projections

Let U be a vector space with subspaces V, W, and assume that $U = V \oplus W$. This means that every vector $u \in U$ can be written uniquely as $u = v + w$ for $v \in V$ and $w \in W$.

In such situations, we can define maps $L : U \longrightarrow V$ and $M : U \longrightarrow W$ via $L(u) = v$ and $L(u) = w$.

Exercises:

(1) Let $U = \mathbb{R}^3$, V the x-axis and W the $y-z$-plane. Show that $U = V \oplus W$.
(2) Let U and W be as above, but this time take V to be the span of $\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)$, that is, the vector space of all vectors of the form $\left(\begin{smallmatrix}a\\a\\0\end{smallmatrix}\right)$. Show that $U = V \oplus W$.
(3) Show that L and M are linear maps.
(4) Show that $M = \mathrm{id} - L$ (that means $M(u) = u - L(u)$ for all $u \in U$).
(5) Show that $\ker L = W$ and $\ker M = V$.
(6) Show that $\mathrm{im}\ L = V$ and $\mathrm{im}\ M = W$.
(7) Show that $L(L(u)) = L(u)$ and $L(M(u)) = 0$.

L is called the projection of U down to V (with respect to the decomposition $U = V \oplus W$; if $U = V' \oplus W$ is another decomposition of U as a direct sum, the projection L' down to V' will be a different linear map).

If U is an inner product space and V some subspace, then $W^\perp = \{u \in U : (u, v) = 0 \text{ for all } v \in V\}$ is called the orthogonal complement of V. It is the set of all vectors in U that are orthogonal to every vector in V. In such a situation we have $U = V \oplus V^\perp$, and the linear map $L : U \longrightarrow V$ defined above is called the orthogonal projection down to V.

In the book (Section 3.5/4.5 in the 7th/8th edition), the map defined by $L(u) = v$ is denoted by $L = \mathrm{proj}_{\,V}(u)$. If you want to be able to compute $\mathrm{proj}_{\,V}(u)$ using inner products, what you need to do is the following:

(1) Choose an orthonormal basis $\{v_1, \dots, v_r\}$ for V;
(2) compute the inner products $c_1 = (u, v_1)$, …, $c_r = (u, v_r)$;
(3) observe that $v = c_1 v_1 + \ldots + c_r v_r$ has the right properties, namely:
    (a) $v \in V$ (obvious).
    (b) $(v, w) = 0$ (all the v_i are orthogonal to W)
    (c) $w = u - v \in W$: since W is the orthogonal complement of V, all you need to do is verify that $(w, v) = 0$ for all $v \in V$. It is enough to check this for a basis of V, i.e., that $(w, v_i) = 0$. Do this!

If we represent L by a matrix P (with respect to some basis, for example the standard basis in $V = \mathbb{R}^n$; recall that the first column of P is just the image of the first basis vector under L), then $L(L(u)) = L(u)$ corresponds to $P^2(u) = P(u)$, that is, to $P^2 = P$. Since $M = \mathrm{id} - L$, the matrix corresponding to M is $I - P$, where I is the identity matrix.