LINEAR ALGEBRA I

THE STORY SO FAR

1. Vector Spaces

Informally, a real **vector space** consists of a set of elements that you can add and multiply by real constants called scalars – the exact set of axioms can be found in any book on linear algebra.

If V is a vector space and if U is a subset of V then we say that U is a **subspace** of V if U is also a vector space. Example: the vector space of polynomials of degree ≤ 2 is a subspace of the vector space of polynomials of degree ≤ 3.

A **linear combination** of vectors $v_1, \ldots, v_n$ is an expression $a_1v_1 + \ldots + a_nv_n$ with real constants $a_1, \ldots, a_n$.

The **span** of vectors $v_1, \ldots, v_n$ from some vector space V is the set of all linear combinations $a_1v_1 + \ldots + a_nv_n$. Theorem: $\mathrm{span}(v_1, \ldots, v_n)$ is a subspace of V.

Vectors $v_1, \ldots, v_n$ are called **linearly dependent** if there is a nontrivial solution of $a_1v_1 + \ldots + a_nv_n = 0$. If $a_1 = \ldots = a_n = 0$ is the only solution, they are called **linearly inpendent**.

2. Examples

(1) Examples of real vector spaces: the set of all vectors $\binom{x}{y}$ with $x, y \in \mathbb{R}$ (similarly for higher "dimension", the set of all polynomials, the set of all polynomials of degree $\leq n$ for some fixed n, the set of all continuous functions $\mathbb{R} \longrightarrow \mathbb{R}$, the set of solutions of the differential equation $y' = y$ on $\mathbb{R}$, and a whole lot more.

(2) Write $\binom{3}{2}$ as a linear combination of $\binom{-1}{2}$ and $\binom{1}{1}$.

From $\binom{3}{2} = a\binom{-1}{2} + b\binom{1}{1}$ we get the linear equations $3 = -a + b$ and $2 = 2a + b$. Solving this system shows $a = -\frac{1}{3}$ and $b = \frac{8}{3}$.

(3) Are the polynomials $x^2 - 1$, $x + 1$, $x^2 + x$ linearly independent in the vector space of polynomials of degree ≤ 2?

Write $a(x^2-1)+b(x+1)+c(x^2+x) = 0$ for scalars $a, b, c \in \mathbb{R}$. This implies $(a+c)x^2+(b+c)x+(b-a)\cdot 1 = 0$. Now a polynomial is the 0-element if and only if all of its coefficients are 0, hence we see that $a+c = 0$, $b+c = 0$ and $b-a = 0$. Eliminating c we get $a-b = 0$ (twice), hence $a = b = 1$ and $c = -1$ is a nonzero solution, i.e., $(x^2-1)+(x+1) = (x^2+x)$ is a nontrivial relation between these polynomials. In particular, $\mathrm{span}(x^2 - 1, x + 1, x^2 + x) = \mathrm{span}(x^2 - 1, x + 1) = \mathrm{span}(x + 1, x^2 + x) = \mathrm{span}(x^2 - 1, x^2 + x)$.

(4) Assume that the vectors v_1, v_2, v_3 are linearly independent in U, and that U is a subspace of V. Are v_1, v_2, v_3 linearly independent in V?

If v_1, v_2, v_3 were linearly dependent in V, then there would be a nontrivial relation $a_1v_1 + a_2v_2 + a_3v_3 = 0$ in V. But this is also a relation in U, so v_1, v_2, v_3 would also be linearly dependent in U, which they are not.

Thus the answer to the question is "yes".