LINEAR ALGEBRA

PROBLEMS

(1) Write a polynomial $f(x) = a_n x^n + \ldots + a_1 x + a_0$ as a linear combination of 1, $x - a$, $(x-a)^2$, ..., $(x-a)^n$ for some real number a.

(2) Show that $M_2 \simeq W_1 \oplus W_2$, where

$$W_1 = \left\{ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix} \right) \middle| \; a, b \in \mathbb{R} \right\}, \text{ and } W_2 = \left\{ \left(\begin{smallmatrix} c & d \\ d & -c \end{smallmatrix} \right) \middle| \; c, d \in \mathbb{R} \right\}.$$

(3) Let A, B, C be $n \times n$-matrices, and assume that A and C are nonsingular. Show that

$$\left(\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \right)$$

is nonsingular, and compute its inverse.

(4) Show that $\text{Tr}(AB) = \text{Tr}(BA)$.

(5) A matrix A is called nilpotent if $A^n = 0$ for some $n \geq 2$. Show that if A is nilpotent, then $I - A$ is nonsingular. Find $(I - A)^{-1}$.

(6) Let V be the vector space of polynomials. Define maps $A, B : V \longrightarrow V$ via

$$A(p(x)) = p'(x), \qquad B(p(x)) = xp(x)$$

(taking the derivative and multiplication by x).
   (a) Show that A and B are linear maps.
   (b) Show that A is onto (im $A = V$) and that $\dim \ker A = 1$.
   (c) Show that $\ker B = 0$ and that B is not onto.
   (d) Show that $AB - BA = I$, where I is the identity map.

(7) Let A be an idempotent square matrix, i.e., assume that $A^2 = A$.
   (a) Show that $I - A$ is idempotent;
   (b) Show that $(2A - I)^2 = I$.
   (c) Show that $A + I$ is invertible, and find $(A + I)^{-1}$.
   (d) Show that $\ker L = \{x - Ax : x \in V\}$, where L is the linear map $L(x) = Ax$.

(8) Let V be an inner product space. Show that

$$||x + y||^2 + ||x - y||2 = 2||x||^2 + 2||y||^2.$$

If x and y are orthogonal, show that

$$||x + y||^2 = ||x||^2 + ||y||^2.$$

(9) Let W be a subspace of the inner product space V. Show that $(W^\perp)^\perp = W$.

1. Hints

(1) Taylor expansion.
(2) Write every 2×2-matrix as a linear combination of elements in W_1 and W_2, and show that $W_1 \cap W_2 = 0$.
(3) Can you do it if $n = 1$, i.e., if A, B and C are just real numbers?
(4) It is ok to do this for 3×3-matrices only.
(5) Recall the geometric series for $\frac{1}{1-x}$. "Plug in" $x = A$. This is not a proof; but it is easy to show that the result is correct.
(6) This should not be too hard.
(7) The first two problems are easy. For the third, use the second.
(8) This is easy.
(9) This is not hard; the only difficulty is finding out exactly what you have to prove. Write down the definitions of $W^{\perp}$ and $(W^{\perp})^{\perp}$.