

LINEAR ALGEBRA

MIDTERM 2, 01.12.2005

- (1) It is your responsibility to write clearly.
- (2) Show all your work. Correct answers without sufficient explanation will not get full credit.
- (3) Your answer should consist of complete sentences.

(1) Complete the following sentences: a linear map $L : U \rightarrow V$ is

- injective (1 – 1) if and only if $\dim \ker L = 0$.
- surjective (onto) if and only if $\dim \operatorname{im} L = \dim V$.
- an isomorphism if and only if $\dim \ker L = 0$ and $\dim \operatorname{im} L = \dim V$.

If L is injective, then $\dim U = \dim \ker L + \dim \operatorname{im} L = \dim \operatorname{im} L \leq \dim V$ (since $\operatorname{im} L \subseteq V$); thus injectivity implies $\dim U \leq \dim V$. On the other hand, the zero map (every vector goes to 0) from \mathbb{R}^2 to \mathbb{R}^3 is clearly not injective, yet $\dim \mathbb{R}^2 < \dim \mathbb{R}^3$.

(2) Let U be a subspace of the inner product space, and let u_1, u_2, u_3 be a basis of U . Show that v_1, v_2, v_3 are pairwise orthogonal vectors in U , where

$$\begin{aligned}v_1 &= u_1, \\v_2 &= u_2 - \frac{(u_2, v_1)}{(v_1, v_1)}v_1, \\v_3 &= u_3 - \frac{(u_3, v_2)}{(v_2, v_2)}v_2 - \frac{(u_3, v_1)}{(v_1, v_1)}v_1.\end{aligned}$$

We need to show that $(v_1, v_2) = (v_1, v_3) = (v_2, v_3) = 0$. This is done as follows:

$$\begin{aligned}(v_1, v_2) &= (v_1, u_2 - \frac{(u_2, v_1)}{(v_1, v_1)}v_1) \\&= (v_1, u_2) - (u_2, v_1) = 0, \\(v_1, v_3) &= (v_1, u_3 - \frac{(u_3, v_2)}{(v_2, v_2)}v_2 - \frac{(u_3, v_1)}{(v_1, v_1)}v_1) \\&= (v_1, u_3) - 0 - (u_3, v_1) = 0, \\(v_2, v_3) &= (v_2, u_3 - \frac{(u_3, v_2)}{(v_2, v_2)}v_2 - \frac{(u_3, v_1)}{(v_1, v_1)}v_1) \\&= (v_2, u_3) - (u_3, v_2) - 0 = 0,\end{aligned}$$

where we have twice used that $(v_1, v_2) = 0$. If you plug the formulas for v_2

and v_3 into (v_2, v_3) , the calculations become a lot more involved. Of course, you also may start with $v_2 = u_2 + av_1$ and determine a from $(v_2, v_1) = 0$.

Remark. If you did something like $\frac{(u_2, v_1)}{(v_1, v_1)}v_1 = \frac{u_2 v_1 v_1}{v_1 v_1} = u_2$, you did not get any credit at all, and you should look up the notion of inner products. NOW.

- (3) Show that three nonzero, pairwise orthogonal vectors v_1, v_2, v_3 in an inner product space are always linearly independent.

Assume that $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$. Then

$$\begin{aligned} 0 &= (v_1, 0) = (v_1, a_1 v_1 + a_2 v_2 + a_3 v_3) \\ &= a_1 (v_1, v_1) + a_2 (v_1, v_2) + a_3 (v_1, v_3) \\ &= a_1 (v_1, v_1). \end{aligned}$$

Since $v_1 \neq 0$, we know that $(v_1, v_1) \neq 0$, and this implies $a_1 = 0$. Similarly, we can derive $a_2 = a_3 = 0$.

Remark. Even if you had no idea what to do: at the very least you could have written down the equation $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$ and explained that you have to show that $a_1 = a_2 = a_3 = 0$. Some had the equation and erased it. WHY?

Also, an inner product is an inner product, and (v_1, v_1) may be written in the form $v_1 \cdot v_1$ if we talk about the standard inner product on \mathbb{R}^n . It is **never** written in the form $(v_1, v_1) = v_1^2$.

- (4) Use the Gram-Schmidt process to find an orthonormal basis of the inner product space P_2 , where $(p, q) = \int_{-1}^1 p(t)q(t)dt$.

We start with the standard basis $\{1, t, t^2\}$ and apply Gram-Schmidt. Since $(1, 1) = 2$, $(t, 1) = (t^2, t) = 0$, and $(t^2, 1) = \frac{2}{3}$, the Gram-Schmidt formulas give the orthogonal basis $\{1, t, t^2 - \frac{1}{3}\}$.

It remains to compute the lengths of these polynomials. We find $(1, 1) = 2$, $(t, t) = \frac{2}{3}$, and $(t^2 - \frac{1}{3}, t^2 - \frac{1}{3}) = \frac{8}{45}$. Thus an orthonormal basis is given by $\{\frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}t, \frac{\sqrt{90}}{2}(t^2 - \frac{1}{3})\}$.

Remark. Of course, you may also start with the basis $\{t^2, t, 1\}$ or even $\{t^2 - t, t + 7, t - 17\}$, but this will not make things any simpler.

- (5) Let $M_2(\mathbb{C})$ be the vector space of 2×2 -matrices with complex entries, and define an inner product on $M_2(\mathbb{C})$ via $(A, B) = \text{Tr}(\overline{B}^T A)$. Find an orthonormal basis for the subspace W of $M_2(\mathbb{C})$ spanned by

$$A = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1+i & 0 \\ 1 & 0 \end{pmatrix}.$$

Gram-Schmidt shows that $\{A, C\}$ is an orthogonal basis of W , where $C = B - \frac{(B,A)}{(A,A)}A$. Now

$$\begin{aligned}(A, A) &= \text{Tr} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} = \text{Tr} \begin{pmatrix} 1 & * \\ * & 2 \end{pmatrix} = 3, \\(A, B) &= \text{Tr} \begin{pmatrix} 1-i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} = \text{Tr} \begin{pmatrix} 1-i & * \\ * & 0 \end{pmatrix} = 1-i, \\(B, A) &= \text{Tr} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1+i & 0 \\ 1 & 0 \end{pmatrix} = \text{Tr} \begin{pmatrix} 1+i & * \\ * & 0 \end{pmatrix} = 1+i, \\(B, B) &= \text{Tr} \begin{pmatrix} 1-i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+i & 0 \\ 1 & 0 \end{pmatrix} = \text{Tr} \begin{pmatrix} 3 & * \\ * & 0 \end{pmatrix} = 3.\end{aligned}$$

Note that $(A, B) = \overline{(B, A)}$ as it should be.

Now we apply Gram-Schmidt. The formula says

$$\begin{aligned}C &= B - \frac{(B,A)}{(A,A)}A = \begin{pmatrix} 1+i & 0 \\ 1 & 0 \end{pmatrix} - \frac{1+i}{3} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2+2i & 1-i \\ 3 & -1-i \end{pmatrix}.\end{aligned}$$

Since, for $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $(D, D) = a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}$, we find $(C, C) = \frac{1}{9}(8+2+9+2) = \frac{21}{9}$. This shows that $\frac{1}{\sqrt{3}}A$ and $\frac{3}{\sqrt{21}}C$ form an orthonormal basis of W .

Remark. Note that, according to our definition of the inner product, we have $(cA, B) = c(A, B)$ and $(A, cB) = \bar{c}(A, B)$ for any $c \in \mathbb{C}$. With this definition, the Gram-Schmidt formulas given in Problem 2 still work: for $C = B - \frac{(B,A)}{(A,A)}A$ we find

$$\begin{aligned}(C, A) &= \left(B - \frac{(B,A)}{(A,A)}A, A \right) = (B, A) - \left(\frac{(B,A)}{(A,A)}A, A \right) \\ &= (B, A) - \frac{(B,A)}{(A,A)}(A, A) = (B, A) - (B, A) = 0.\end{aligned}$$

Similarly, we find

$$\begin{aligned}(A, C) &= \left(A, B - \frac{(B,A)}{(A,A)}A \right) = (A, B) - \left(\frac{(B,A)}{(A,A)}A, A \right) \\ &= (A, B) - \frac{\overline{(B,A)}}{(A,A)}(A, A) \\ &= (A, B) - \overline{(B, A)} = (A, B) - (A, B) = 0.\end{aligned}$$

Thus everything is fine as long as you stick to the formulas I gave you. If, however, you replace (B, A) by (A, B) , or the definition $(A, B) = \text{Tr}(\overline{B}^T A)$ by $(A, B) = \text{Tr}(B^T \overline{A})$, then you will end up getting nonsense (unless you do both: here two wrongs do make a right).

(6) Compute the determinant

$$\begin{vmatrix} ab & ac & 1 & bc \\ 1 & a & 0 & a^2 \\ 1 & b & 0 & b^2 \\ 1 & c & 0 & c^2 \end{vmatrix}.$$

Developing with respect to the third column shows that this determinant is equal to

$$\begin{aligned} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \\ &= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \\ &= (b-a)(c-a)(c-b). \end{aligned}$$

Remark. In cases where both you and your neighbor had a term bc (instead of bc^2) in your formula, I have allowed myself to take off a few more points for calculational errors than I usually do.

- (7) Consider the map $L : P_2 \rightarrow P_2$ defined by $L(p) = p - tp'$. Compute $\ker L$. What is $\dim \operatorname{im} L$?

We have $L(a + bt + ct^2) = a + bt + ct^2 - t(b + 2ct) = a - ct^2$. Thus $\ker L = \{a + bt + ct^2 : a = c = 0\} = \{bt : b \in \mathbb{R}\}$. Thus $\dim \ker L = 1$, hence $\dim \operatorname{im} L = 2$.

Alternatively, $L(p) = a - ct^2$ shows that the image equals the span of $\{1, t^2\}$, and this again shows that $\dim \operatorname{im} L = 2$.

Remark. If you wonder why you have not received full credit for the correct answer, check what point (2) on page 1 says.

- (8) a) For which values of c does $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = x_1y_1 + x_1y_2 + x_2y_1 + cx_2y_2$ define an inner product on \mathbb{R}^2 ? Linearity and symmetry are easy to check.

It remains to find out whether $(v, v) \geq 0$. Now $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1^2 + 2x_1x_2 + cx_2^2 = (x_1 + x_2)^2 + (c-1)x_2^2$. This is nonnegative whenever $c \geq 1$. If $c = 1$, then $(v, v) = 0$ for the nonzero vector $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, so the formula does not define an inner product if $c = 1$. If $c > 1$ and $(x_1 + x_2)^2 + (c-1)x_2^2 = 0$, then clearly $x_2 = 0$ and $x_1 = -x_2 = 0$.

- b) Find a vector orthogonal to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with respect to this inner product (for general c).

We have $\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = 2y_1 + (2+c)y_2 = 0$ e.g. for $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2+c \\ -2 \end{pmatrix}$.

Remark. When I write “for general c ”, there are various things you may do. What you may not do is start your answer by saying “We may take $c = 1$ ”. This is not allowed even if your neighbors to the left and to the right do it.